# Isomorphisms

Def. Let $V$ and $W$ be vector spaces, and let $T: V \to W$ be linear. A function $U: W \to V$ is said to be the **inverse of $\boldsymbol{T}$** if $TU = I_W$ and $UT = I_V$ . If $T$ has an inverse, then $T$ is said to be **invertible**.

If $T$ is invertible then the inverse of $T$ is unique and denoted $T^{-1}$.

Linear transformations are special cases of functions. A function is invertible if and only if it is one-to-one and onto. Thus we have:

Theorem: Let $T: V \to W$ be a linear transformation where $\dim(V) = \dim(W)$ (both finite). Then $T$ is invertible if and only if $Rank(T) = \dim(V)$.

We saw earlier that when $\dim(V) = \dim(W)$ (both finite) then $Rank(T) = \dim(V)$ is equivalent to $T$ being one-to-one and onto.

The following holds for invertible functions $T$ and $U$.

1. $(TU)^{-1} = U^{-1}T^{-1}$
2. $(T^{-1})^{-1} = T$; thus $T^{-1}$ is invertible.

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(< a_1, a_2 >) =< a_1 + 2a_2,\ a_1 + a_2 >$ using the standard ordered basis for $\mathbb{R}^2$. Show that $T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$T^{-1}(< b_1, b_2 >) =< -b_1 + 2b_2,\ b_1 - b_2 >$ is the inverse of $T$.

$$T^{-1}T(< a_1, a_2 >) = T^{-1}(< a_1 + 2a_2,\ a_1 + a_2 >)$$

$$=< -(a_1 + 2a_2) + 2(a_1 + a_2),\ (a_1 + 2a_2) - (a_1 + a_2) >$$

$$=< a_1, a_2 > .$$

$$TT^{-1}(< b_1, b_2 >) = T(< -b_1 + 2b_2,\ b_1 - b_2 >)$$

$$=< (-b_1 + 2b_2) + 2(b_1 - b_2),\ (-b_1 + 2b_2) + (b_1 - b_2) >$$

$$=< b_1, b_2 >.$$

Thus $T$ and $T^{-1}$ are inverses of eachother.

Notice that if we represent $T$ and $T^{-1}$ in the standard ordered basis $B$ we get:

$$[T]_B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad [T^{-1}]_B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

and

$$[T]_B[T^{-1}]_B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$[T^{-1}]_B[T]_B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Theorem: Let $V$ and $W$ be vector spaces and $T: V \to W$ be linear and invertible. Then $T^{-1}$ is linear.

Proof: Let $w_1, w_2 \in W$ and $c \in \mathbb{R}$.

Since $T$ is one-to-one and onto there exist unique vectors $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$ and thus

$T^{-1}(w_1) = v_1$ and $T^{-1}(w_2) = v_2$.

Therefore we have:

$$
\begin{aligned}
T^{-1}(cw_1 + w_2) &= T^{-1}(cT(v_1) + T(v_2)) \\
&= T^{-1}(T(cv_1 + v_2)) \\
&= cv_1 + v_2 \\
&= cT^{-1}(w_1) + T^{-1}(w_2)
\end{aligned}
$$

and $T^{-1}$ is linear.

Def. Let $A$ be an $n \times n$ matrix. Then $A$ is **invertible** if there exists an $n \times n$ matrix $B$ such that $AB = BA = I$.

Theorem: Let $V$ and $W$ be finite dimensional vector spaces with ordered bases $B_1$ and $B_2$. Let $T: V \to W$ be linear. Then $T$ is invertible if and only if $[T]_{B_1}^{B_2}$ is invertible. Furthermore $[T^{-1}]_{B_2}^{B_1} = \left([T]_{B_1}^{B_2}\right)^{-1}$.

Proof: Suppose $T$ is invertible.

Then $T$ is one-to-one and onto thus $N(T) = 0$ and $Rank(T) = \dim(V)$.

Let $n = \dim(W)$.

$[T]_{B_1}^{B_2}$ is an $n \times n$ matrix.

$T^{-1}: W \to V$ satisfies $TT^{-1} = I_W$ and $T^{-1}T = I_V$.

Thus we have:

$$I_n = [I_V]_{B_1} = [T^{-1}T]_{B_1} = [T^{-1}]_{B_2}^{B_1}[T]_{B_1}^{B_2}.$$

Similarly, we have $[T]_{B_1}^{B_2}\,[T^{-1}]_{B_2}^{B_1} = I_n$.

So $[T]_{B_1}^{B_2}$ is invertible and $\left([T]_{B_1}^{B_2}\right)^{-1} = [T^{-1}]_{B_2}^{B_1}$.

Now suppose that $A = [T]_{B_1}^{B_2}$ is invertible.

Then there is an $n \times n$ matrix $C$ such that $AC = CA = I$.

There exists a $U \in \mathcal{L}(W, V)$ such that

$$U(w_j) = \sum_{i=1}^{n} C_{ij} v_i \quad \text{for } 1 \le j \le n,$$

where $B_1 = \{v_1, \dots, v_n\}$ and $B_2 = \{w_1, \dots, w_n\}$ are ordered bases for $V$ and $W$. Thus $[U]_{B_2}^{B_1} = C$.

To see that $U = T^{-1}$ note that:

$$[UT]_{B_1} = [U]_{B_2}^{B_1}[T]_{B_1}^{B_2} = CA = I_n = [I_V]_{B_1},$$

So $UT = I_V$. Similarly, $TU = I_W$.

Corollary: Let $V$ be a finite dimensional vector space with ordered basis $B$ and let $T: V \to V$ be linear. Then $T$ is invertible if and only if $[T]_B$ is invertible. Furthermore $[T^{-1}]_B = ([T]_B)^{-1}$.

Def. Let $V$ and $W$ be vector spaces. We say ***V* is isomorphic to *W*** if there exists a linear transformation $T: V \to W$ that is invertible. In this case $T$ is called an **isomorphism**.

Ex. Show that $T: \mathbb{R}^3 \to P_2(\mathbb{R})$ by $T(< a_1, a_2, a_3 >) = a_1 + a_2x + a_3x^2$ is an isomorphism.

We have already seen that $T$ is linear.

$\dim(\mathbb{R}^3) = \dim\big(P_2(\mathbb{R})\big) = 3$ and $N(T) = \{0\}$ so $T$ is one-to-one and onto.

Thus $T$ is invertible and an isomorphism.

The inverse map is:

$$T^{-1}(a_1 + a_2x + a_3x^2) =< a_1 , a_2, a_3 >.$$

A straight forward calculation shows that :

$$T^{-1}T = I_{\mathbb{R}^3}$$

$$TT^{-1} = I_{P_2(\mathbb{R})}.$$

Theorem: Let $V$ and $W$ be finite dimensional vector spaces. Then $V$ is isomorphic to $W$ is and only if $\dim(V) = \dim(W)$.

Proof: Suppose $V$ is isomorphic to $W$ and $T: V \to W$ is an isomorphism.

Since $T$ is one-to-one and onto $\dim(V) = \dim(W)$.

Now let's assume that $\dim(V) = \dim(W)$ and show $V$ is isomorphic to $W$.

Let $B_1 = \{v_1, \dots, v_n\}$, $B_2 = \{w_1, \dots, w_n\}$ be ordered bases for $V$ and $W$ respectively.

We can define a linear transformation $T: V \to W$ by $T(v_i) = w_i,\ 1 \le i \le n$.

$$R(T) = span\{T(v_1), \dots, T(v_n)\}$$
$$= span\{w_1, \dots, w_n\}$$
$$= W.$$

So $T$ is onto.

Since $\dim(V) = \dim(W)$, $T$ must also be one-to-one.

Hence $T$ is an isomorphism.

Corollary: Every vector space $V$ with $\dim(V) = n$ is isomorphic to $\mathbb{R}^n$.

Ex. By the previous corollary, $M_{n\times n}(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n^2}$ since $\dim\big(M_{n\times n}(\mathbb{R})\big) = n^2$.

Ex. Find an isomorphism from $S_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle| \, a, b, d \in \mathbb{R} \right\}$ to $\mathbb{R}^3$.

Let $T: S_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} a & b \\ b & d \end{bmatrix}\right) =< a, b, d >$.

We need to show that $T$ is linear, one-to-one, and onto.

To show that $T$ is linear let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$, and $c \in \mathbb{R}$.

$$T(cA + B) = T(\begin{bmatrix} ca_{11} & ca_{12} \\ ca_{12} & ca_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix})$$

$$= T(\begin{bmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} \\ ca_{12} + b_{12} & ca_{22} + b_{22} \end{bmatrix})$$

$$=< ca_{11} + b_{11},\ ca_{12} + b_{12},\ ca_{22} + b_{22} >$$

$$= c < a_{11}, a_{12}, a_{22} > +< b_{11}, b_{12}, b_{22} >$$

$$= cT(A) + T(B).$$

So $T$ is linear.

To show that $T$ is one-to-one we show that $N(T) = \{0\}$.

$$T(A) =< a_{11}, a_{12}, a_{22} >=< 0,0,0 > \implies \quad a_{11} = 0,\ a_{12} = 0,\ a_{22} = 0.$$

Thus $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $N(T) = \{0\}$.

To show that $T$ is onto, take any element $< a, b, d >\in \mathbb{R}^3$ and let's show we can find $A \in S_{2\times 2}(\mathbb{R})$ such that $T(A) =< a, b, d >$.

Let $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, then $T(A) =< a, b, d >$, and $T$ is onto.

Thus $T$ is an isomorphism of $S_{2\times 2}(\mathbb{R})$ and $\mathbb{R}^3$.