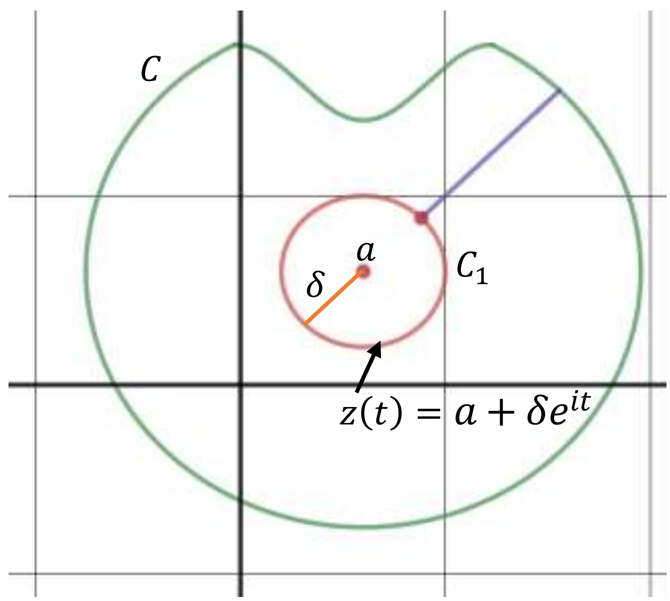
Cauchy's Integral Formula

Cauchy's integral formula shows that the values of an analytic function f on the boundary of a closed contour C determine the values of f interior to C.

Theorem (Cauchy's Integral Formula) Let $f(z)$ be analytic interior to and on a simple closed contour C. Then at any interior point $z = a$

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz.$$

Outline of Proof: Start by making a crosscut from C to a circle of radius δ around a. Call that circle C_1.



By Cauchy's theorem we have

$$\oint_C \frac{f(z)}{z-a} dz = \oint_{C_1} \frac{f(z)}{z-a} dz.$$

Now we write:

$$\oint_{C_1} \frac{f(z)}{z-a} dz = f(a) \oint_{C_1} \frac{1}{z-a} dz + \oint_{C_1} \frac{f(z)-f(a)}{z-a} dz.$$

By parametrizing the circle of radius δ around a, $z(t) = a + \delta e^{it}$, we get:

$$\oint_{C_1} \frac{1}{z-a} dz = 2\pi i.$$

Since $f(z)$ is continuous we know for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|z - a| < \delta$ then $|f(z) - f(a)| < \epsilon$. Thus we can say:

$$\left|\oint_{C_1} \frac{f(z)-f(a)}{z-a} dz\right| \leq \oint_{C_1} \frac{|f(z)-f(a)|}{|z-a|} |dz|$$

$$\leq \oint_{C_1} \frac{\epsilon}{\delta} |dz| = (2\pi\delta)\left(\frac{\epsilon}{\delta}\right) = 2\pi\epsilon.$$

Since ϵ is any positive number $\Longrightarrow \oint_{C_1} \frac{f(z)-f(a)}{z-a} dz = 0.$

Thus $$\oint_C \frac{f(z)}{z-a} dz = 2\pi i(f(a))$$

or $$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz.$$

We can now show that if $f(z)$ is analytic (i.e. has one derivative) then it must have an infinite number of derivatives and we can find a formula for them.

Theorem (We will also call this Cauchy's Integral Formula): Let $f(z)$ be analytic interior to and on a simple closed contour C, then $f^{(k)}(z)$, $k = 1,2, \ldots$ exists in the domain D interior to C and

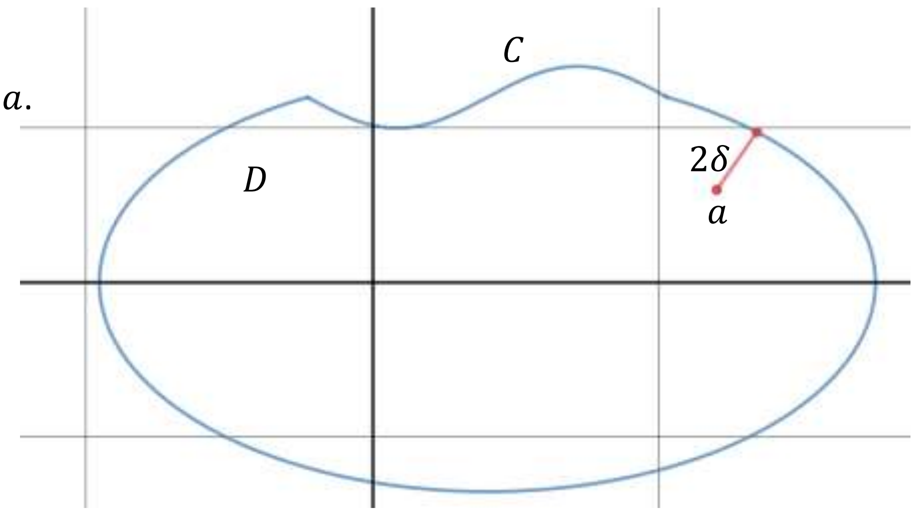
$$f^{(k)}(a) = \frac{k!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{k+1}} dz.$$

Cauchy's Integral Formula tells us that if a complex function has one derivative in a domain D bounded by a simple closed contour, it has an in infinite number of derivatives (later we will also see that the Taylor series of $f(z)$ will also converge to the function $f(z)$ in D) in D. This is very different from real valued functions. For functions of a real variable you can have a function with one derivative, but not two derivatives, or two derivatives but not three derivatives, and so on.

Proof: Let's start by proving the formula for $k = 1$.

Let $2\delta = \min|z-a|, \ \ z \in C$,

i.e., z is the closest point on C to a.



Using Cauchy's Integral Formula we can say:

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\left(\frac{1}{h}\right)\oint_C \left(\frac{f(z)}{z-(a+h)} - \frac{f(z)}{z-a}\right)dz$$

$$= \frac{1}{2\pi i}\left(\frac{1}{h}\right)\oint_C \ f(z)\left(\frac{h}{(z-(a+h))(z-a)}\right)dz$$

$$= \frac{1}{2\pi i}\oint_C \ f(z)\left(\frac{1}{(z-(a+h))(z-a)}\right)dz.$$

Notice that:

$$\frac{1}{(z-(a+h))(z-a)} = \frac{1}{(z-a)^2} + \frac{h}{(z-a)^2\big(z-(a+h)\big)}\,.$$

So we have:

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{(z-a)^2}\right)dz + \frac{h}{2\pi i}\oint_C f(z)\left(\frac{1}{(z-a)^2\big(z-(a+h)\big)}\right)dz.$$

So we just need to show that:

$$\lim_{h\to 0}\frac{h}{2\pi i}\oint_C f(z)\left(\frac{1}{(z-a)^2\big(z-(a+h)\big)}\right)dz = 0.$$

If we choose h so that $|h| < \delta$, then we have by the triangle inequality:

$$|z-(a+h)| \geq |z-a| - |h| > 2\delta - \delta = \delta.$$

Since $f(z)$ is continuous on C, $f(z)$ is bounded in C. So there is a real number M such that $|f(z)| \leq M$ for all $z \in C$.

Thus we can say:

$$\left|\frac{f(z)}{(z-a)^2\big(z-(a+h)\big)}\right| \leq \frac{M}{(2\delta)^2\delta}.$$

Hence we have:

$$0 \leq \left|\frac{h}{2\pi i}\oint_C f(z)\left(\frac{1}{(z-a)^2\big(z-(a+h)\big)}\right)dz\right| \leq \frac{M|h|}{(2\delta)^2\delta}\oint_C |dz| = \frac{M|h|}{(2\delta)^2\delta}L.$$

Where L is the length of C.

Now as h goes to 0, the right hand side goes to 0.

Thus by the squeeze theorem, $\lim_{h\to 0}\frac{h}{2\pi i}\oint_C f(z)\left(\frac{1}{(z-a)^2\big(z-(a+h)\big)}\right)dz = 0.$

Thus we have: $f'(a) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz.$

Repeating this argument gives the formula for higher order derivatives.

Theorem: let $f(z) = u(x, y) + iv(x, y)$ be analytic in D. Then all partial derivatives of u and v, of all orders, are continuous in D.

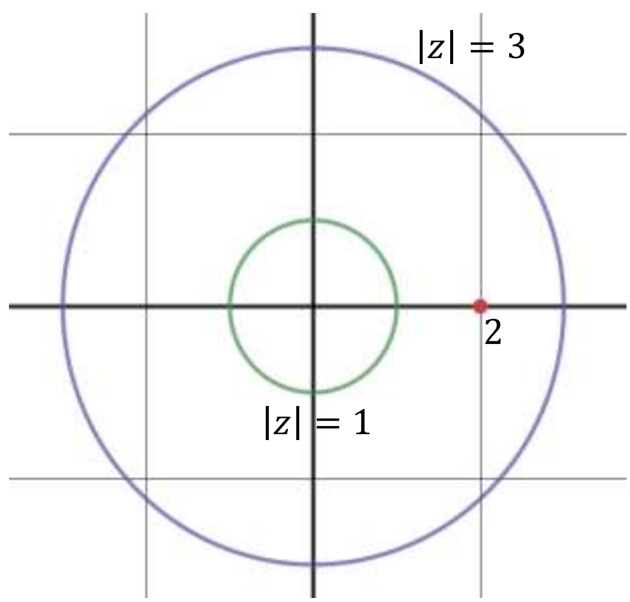
This follow directly because $f^{(k)}(z)$ exists for all $k = 1,2,3, \ldots$ and because:

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Cauchy's integral formula gives us a way to evaluate many integrals around a simple closed contour without parametrizing the curve.

Ex. Evaluate $\oint_C \frac{e^z}{z-2} dz$ where

a. C is the circle $|z| = 3$
b. C is the circle $|z| = 1$.



a. By Cauchy's Integral Formula:

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz,$$ when a is inside of C.

In this example, $a = 2$, $f(z) = e^z$ and 2 is inside the circle $|z| = 3$.

$$f(2) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-2}\,dz,$$ where $f(z) = e^z$; So

$$e^2 = \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz$$ or

$$2\pi e^2 i = \oint_C \frac{e^z}{z-2}dz.$$

Note: This integral can also be evaluated by doing the following:

$$\oint_C \frac{e^z}{z-2}dz = \oint_C \frac{e^{(z-2)}e^2}{z-2}dz = e^2\oint_C \frac{e^{(z-2)}}{z-2}dz.$$

Now make a crosscut into a circle, C_1, of radius $\delta < 1$.

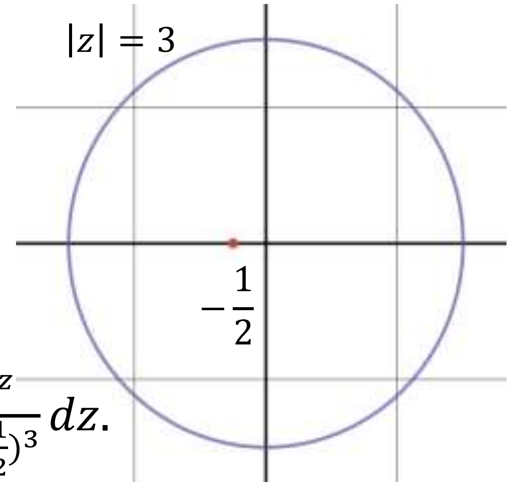
Using Cauchy's theorem (see previous section) we get:

$$e^2\oint_C \frac{e^{(z-2)}}{z-2}dz = e^2\oint_{C_1} \frac{e^{(z-2)}}{z-2}dz.$$

Now make the substitution $w = z - 2,\ \ dw = dz$, and use a power series for e^w.

b. $a = 2$ is outside the circle C is the circle $|z| = 1$, thus $\frac{e^z}{z-2}$ is analytic inside the circle $|z| = 1$. So by Cauchy's Theorem: $\oint_C \frac{e^z}{z-2}dz = 0.$

Ex. Evaluate $\oint_C \frac{e^{2z}}{(2z+1)^3}dz$, where C is the circle $|z| = 3$.



Cauchy's Integral Formula for derivatives is:

$$f^{(k)}(a) = \frac{k!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{k+1}}dz$$

$$\oint_C \frac{e^{2z}}{(2z+1)^3}dz = \oint_C \frac{e^{2z}}{8(z+\frac{1}{2})^3}dz = \frac{1}{8}\oint_C \frac{e^{2z}}{(z+\frac{1}{2})^3}dz.$$

$$f^{(2)}(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz\ .$$

So if $f(z) = e^{2z}$; $\quad a = -\frac{1}{2}$.

$f'(z) = 2e^{2Z}$

$f''(z) = 4e^{2z}$ and $f''\left(-\frac{1}{2}\right) = 4e^{-1}$.

$$f''\left(-\frac{1}{2}\right) = \frac{2!}{2\pi i}\oint_C \frac{e^{2z}}{(z+\frac{1}{2})^3}dz = \frac{1}{\pi i}\oint_C \frac{e^{2z}}{(z+\frac{1}{2})^3}dz$$

$$4e^{-1} = \frac{1}{\pi i}\oint_C \frac{e^{2z}}{(z+\frac{1}{2})^3}dz$$

$$4\pi i e^{-1} = \oint_C \frac{e^{2z}}{(z+\frac{1}{2})^3}dz\ ;$$ so

$$\frac{\pi i}{2} e^{-1} = \frac{1}{8} \oint_C \frac{e^{2z}}{(z+\frac{1}{2})^3} dz = \oint_C \frac{e^{2z}}{(2z+1)^3} dz.$$

Note: This integral can also be evaluated by:

$$\oint_C \frac{e^{2z}}{(2z+1)^3} dz = e^{-1} \oint_C \frac{e^{(2z+1)}}{(2z+1)^3} dz; \text{ let } w = 2z + 1.$$

Now use a crosscut to a circle around $w = 0$, (i.e. $z = -\frac{1}{2}$) and a power series for e^w.

Theorem: Let C be a circle of radius R around $z = a$. If $f(z)$ is analytic inside and on C then:

$$\left|f^{(n)}(a)\right| \le \frac{n!M}{R^n}$$

where $|f(z)| \le M$ for $z \in C$.

Proof:

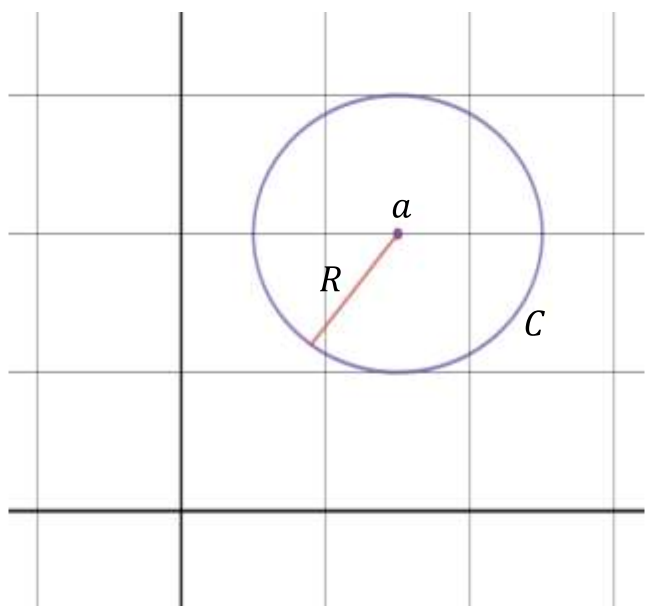
C is the circle, $|z - a| = R$.

Since $f(z)$ is continuous on C

(since it's differentiable on C)

we know that there is an $M \in \mathbb{R}$

such that $|f(z)| \le M$ for $z \in C$.



By Cauchy's Integral Theorem for

derivatives we have:

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

Which means:

$$|f^{(n)}(a)| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \leq \frac{n!}{2\pi}\oint_C \frac{|f(z)|}{|z-a|^{n+1}}|dz|$$

$$\leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}\oint_C |dz|;$$

But $\oint_C |dz|$=Arclength of $C = 2\pi R$, so

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}(2\pi R) = \frac{n!M}{R^n}.$$

Recall that an entire function is a function that is analytic in the complex plane (excluding the point at ∞).

Theorem (Liouville) If $f(z)$ is entire and bounded in the complex plane then $f(z) =$constant.

Proof: Using the inequality $\left|f^{(n)}(a)\right| \leq \frac{n!M}{R^n}$ when $n = 1$ we have:

$|f'(a)| \leq \frac{M}{R}$; where $|f(z)| \leq M$ for $z \in C$, a circle of radius R around $z = a$, for any point $a \in \mathbb{C}$.

But $f(z)$ is bounded on the complex plane, so there is an $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for $z \in \mathbb{C}$.

Thus for any point $a \in \mathbb{C}$, $|f'(a)| \leq \frac{M}{R}$ for any circle of radius R around $z = a$ (that is, the same M works for any point $z = a$ and circle of radius R).

So as R goes to infinity $\frac{M}{R}$ goes to 0.

Thus $f'(a) = 0$ for any point $a \in \mathbb{C}$. Thus $f(z) =$constant.

Again, this is a difference between complex functions and functions of a real variable. There are many examples of real valued functions that are differentiable on $\mathbb{R}$ (or $\mathbb{R}^n$), bounded, and are not equal to constant functions

(e.g. $f(x) = sinx, \quad f(x) = \frac{1}{1+x^2}$, etc.)

Why don't $f(z) = sinz$ or $f(z) = \frac{1}{1+z^2}$ contradict Liouville's theorem?

(Answer: They are not bounded on $\mathbb{C}$).

Corollary (Fundamental Theorem of Algebra): Any m^{th} degree polynomial, $P(z)$, $m \geq 1$ has at least one root (i.e. a point $a \in \mathbb{C}$, such that $P(a) = 0$).

This actually implies that any m^{th} degree polynomial has exactly m roots.

Proof: Proof by contradiction. Assume that $P(z)$, a polynomial of degree $m \geq 1$, is never 0 and hence does not have a root.

Let $R(z) = \frac{1}{P(z)}$.

Then $R(z)$ is analytic in $\mathbb{C}$. Also, as $|z| \to \infty$, $P(z) \to \infty$, and hence $R(z) \to 0$. Thus $R(z)$ is bounded in $\mathbb{C}$.

By Liouville's theorem $R(z) = \frac{1}{P(z)}$ must be a constant.

Thus $P(z) =$constant.

This contradicts that $P(z)$ is a polynomial of degree $m \geq 1$.

Thus $P(z)$ must have a root.

Cauchy's theorem says if $f(z)$ is analytic inside and including a simple closed contour C, then $\oint_C \; f(z)dz = 0$. Morera's theorem shows the converse is true.

Morera's Theorem: If $f(z)$ is continuous in a simply connected domain D and if $\oint_C \; f(z)dz = 0$ for every simple closed contour C lying in D, then $f(z)$ is analytic in D.

Proof: We had a theorem that said if $f(z)$ is continuous in a simply connected domain D and if $\oint_C \; f(z)dz = 0$ for every simple closed contour C lying in D, then there exists a function $F(z)$, analytic in D, such that $F'(z) = f(z)$.

From Cauchy's Integral Formula we know that if a function $F(z)$ has a derivative (i.e. is analytic) then it has an infinite number of derivatives and hence all derivatives of $F(z)$ are analytic.

Since $f(z) = F'(z)$, $f(z)$ is analytic.

Ex. Use Cauchy's Integral formula to show that $f(z) = \frac{cosz}{z}$ is not analytic inside a circle of radius R.

By Cauchy's Integral Formula: $1 = \cos(0) = \frac{1}{2\pi i} \oint_C \frac{cosz}{z} dz$ for a closed curve inside a circle of radius R thus $\oint_C \frac{cosz}{z} dz \neq 0$.

Hence $f(z) = \frac{cosz}{z}$ is not analytic inside a circle of radius R by Cauchy's theorem.

By Cauchy's Integral Theorem we know if $f(z)$ is analytic on and inside C

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz.$$

In particular, if we take C to be a circle of radius r around $z = a$:

$$z(\theta) = a + re^{i\theta}; \quad 0 \leq \theta \leq 2\pi, \quad dz = ire^{i\theta} d\theta$$

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta$$

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta.$$

That is, the value of f at $z = a$ is the average value of f around any circle centered at $z = a$.

If we multiply both sides by $\int_0^R rdr$ we get

$$f(a)\int_0^R rdr = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}\right)d\theta \int_0^R rdr$$

$$f(a)\left(\frac{R^2}{2}\right) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^R f\left(a + re^{i\theta}\right)rdrd\theta$$

$$f(a) = \frac{1}{\pi R^2}\iint_D \ f\left(a + re^{i\theta}\right)dA$$

where D is the disk of radius R with center at $z = a$.

Thus $f(a)$ is also the average value of $f(z)$ over a disk of any radius R with center at $z = a$.

We will now use this result to prove:

Theorem (Maximum Modulus Principal):

1. If $f(z)$ is analytic in a domain D, then $|f(z)|$ cannot have a maximum inside D unless $f(z)$=constant.
2. If $f(z)$ is analytic in a bounded region D and $|f(z)|$ is continuous in a closed region $\overline{D}$, then $|f(z)|$ assumes its maximum on the boundary of the region.

Proof: 1. Let's show if $a \in D$ and $|f(z)| \leq |f(a)|$ for all $z \in D$ then $f(z)$ is a constant function.

Choose any disk, D_0, of radius R around $z = a$ such that $D_0 \subseteq D$.

Let $z = a + re^{i\theta}; \quad 0 \le \theta \le 2\pi$ then

$$f(a) = \frac{1}{\pi R^2} \iint_{D_0} f\left(a + re^{i\theta}\right) dA \quad \text{so we have:}$$

$$|f(a)| \le \frac{1}{\pi R^2} \iint_{D_0} \left|f\left(a + re^{i\theta}\right)\right| dA \le \frac{1}{\pi R^2} \iint_{D_0} |f(a)| dA$$

since $|f(z)| \le |f(a)|$ for all $z \in D$.

But since $|f(a)|$ is a constant we have:

$$|f(a)| \le \frac{1}{\pi R^2} \iint_{D_0} |f(a)| dA = |f(a)|.$$

But that means that $\left|f\left(a + re^{i\theta}\right)\right| = |f(z)| = |f(a)|$ for all $z \in D_0$, otherwise we would have a strict inequality:

$$\frac{1}{\pi R^2} \iint_{D_0} \left|f\left(a + re^{i\theta}\right)\right| dA < \frac{1}{\pi R^2} \iint_{D_0} |f(a)| dA.$$

Thus $|f(z)|$ is a constant on D_0.

The C-R equations imply that an analytic function $f(z)$ whose modulus, $|f(z)|$, is constant must be a constant function.

Now let's show that if D is a bounded region and $|f(z)|$ is continuous on the closed region $\overline{D}$, then $|f(z)|$ assumes its maximum on the boundary of D.

It is a theorem in functions of a real variable that a continuous function on a closed and bounded set (i.e. compact set) in $\mathbb{R}^2$ must take on its maximum and minimum values.

Hence $|f(z)|$ must achieve its maximum value on the boundary of D_0(because it can't have it's maximum inside D by part 1 unless it's a constant, in which case it still takes on its maximum on the boundary).

If $f(z)$ is analytic and non-zero in a region D then $|f(z)|$ has a minimum value on D and it gets achieved on the boundary of D. This can be proved by applying the maximum modulus theorem to $g(z) = \frac{1}{f(z)}$.

The maximum modulus principal also applies to the real and imaginary parts of an analytic function as well as harmonic functions.

Ex. Find the maximum value of $|f(z)|$ on the unit disk, $|z| \leq 1$, for

$$f(z) = e^{(z^2)}.$$

$$z = x + iy \quad \Rightarrow \quad z^2 = (x+iy)(x+iy) = x^2 - y^2 + 2xyi$$

thus

$$\left|e^{(z^2)}\right| = \left|e^{(x^2-y^2)}e^{2xyi}\right| = e^{(x^2-y^2)}.$$

By the maximum modulus principal, since $f(z) = e^{(z^2)}$ is analytic on $|z| \leq 1$, $\left|e^{(z^2)}\right|$ must take on its maximum on the boundary of the unit disk, $|z| = 1$, or the unit circle, $x^2 + y^2 = 1$.

$e^{(x^2-y^2)}$ will have its maximum value when $y^2 = 0$, i.e. $y = 0$ and $x = \pm 1$.

So the maximum value of $|f(z)| = e^{[(\pm 1)^2 - 0]} = e$ and it occurs at z=± 1.