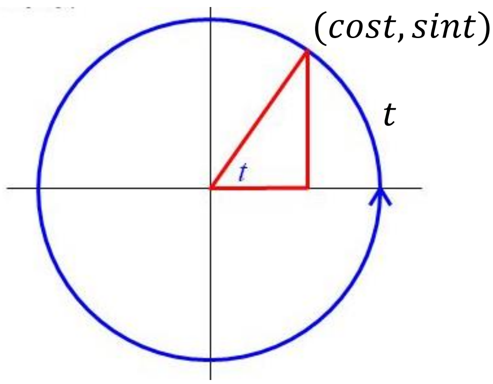
Curves is $\mathbb{R}^2$ and $\mathbb{R}^3$

We are going to study functions where the domain is a subset of the real numbers and the range is points in $\mathbb{R}^3$, i.e. a vectors. As we saw earlier these are called vector valued functions. You've already seen this with parametric equations.

Ex. $x = \cos t$ $\quad$ This is really $\vec{r}: \mathbb{R} \to \mathbb{R}^2, \;\; t \to (\cos t, \sin t)$, so we
$y = \sin t$ $\quad$ could think of this as a vector valued function in $\mathbb{R}^2$,
$\vec{r}(t) = < \cos t, \sin t >$.



In $\mathbb{R}^3$ we will represent a vector valued function by:

$$\vec{r}(t) = < f(t), g(t), h(t) > = f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}.$$

Ex. Let $\vec{r}(t) = < t^2, \; \sqrt{t^2 - 9}, \; \ln t >$, find the domain of $\vec{r}$.

The domain is all values t where $\vec{r}(t)$ is defined:

The domain of t^2 is all real numbers,
The domain of $\sqrt{t^2 - 9}$ is $|t| \geq 3$,
The domain of $\ln t$ is $t > 0$.
So the domain of $\vec{r}$ is the intersection of these 3 sets, $t \geq 3$ (i.e. $[3, \infty)$).

Def. If $\vec{r}(t) = < f(t),\ g(t),\ h(t) >$, then:

$$\lim_{t\to a} \vec{r}(t) = < \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) >$$

(provided the limits of the components exist).

Def. $\vec{r}(t)$ is **continuous at $t = a$** if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$ (i.e. it's continuous if all of the components are continuous.).

We can think of this vector valued function on a subset of $\mathbb{R}$ as a curve in $\mathbb{R}^3$ defined by:

$$\vec{r}(t) = < f(t),\ g(t),\ h(t) >$$

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t).$$

Ex. Describe the curve defined by: $\vec{r}(t) = < 3 - t,\ 2t + 1,\ -3t >$.

$x = 3 - t$
$y = 1 + 2t$
$z = -3t$

The parametric form of a line through $(3, 1, 0)$ with direction vector $< -1, 2, -3 >$.

Ex. Sketch the curve given by $x = -cost, \quad y = 3sint; \quad 0 \le t \le 2\pi.$

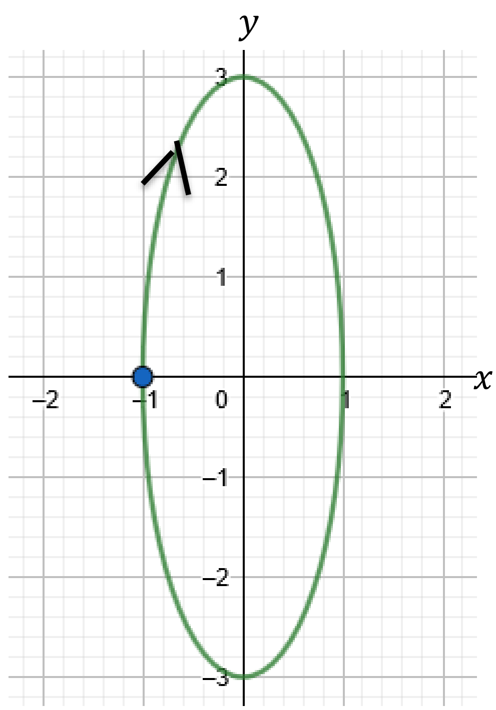
Notice that $\quad x^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1.$

$x^2 + \left(\frac{y}{3}\right)^2 = 1$ is an ellipse with the y axis as the major axis.

The curve starts at $t = 0, \quad x = -\cos(0) = -1, \quad y = 3\sin(0) = 0.$

As t increases from 0, $y = 3sint$ is initially positive.

So the curve starts at $(-1,0)$ and then moves into the second quadrant.
Therefore, this curve moves clockwise around the ellipse from $(-1,0)$.



Ex. Sketch the curve whose vector equation is:

$$\vec{r}(t) = < 4\sin t\,, 4\cos t\,, t > = 4\sin t\,\vec{\imath} + 4\cos t\,\vec{\jmath} + t\vec{k}.$$

Notice:

$x = 4\sin t\,,\quad y = 4\cos t,\quad z = t$ and

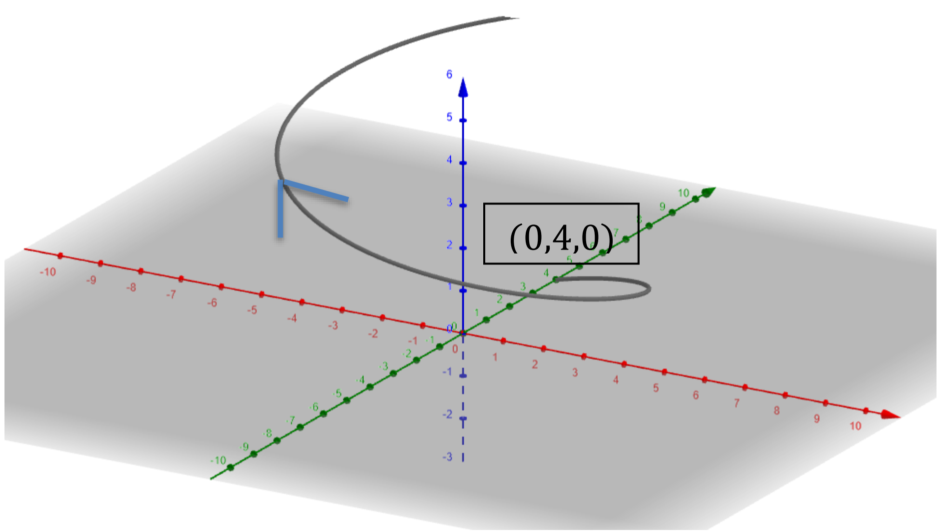
$x^2 + y^2 = 16\sin^2 t + 16\cos^2 t = 16.$

So the curve lies on the cylinder of radius 4
whose axis is the z-axis.

At $t = 0$ we're at $\vec{r}(0) =< 4\sin 0\,, 4\cos 0\,, 0 > =< 0,4,0 >$.

This is a helix.

Which direction does the curve move as t increases?
As t increases, $z = t$ increases, So the curve looks like:



Ex. Find a vector function for the line segment between $P(-3, 2, -1)$ and $Q(0, -2, 3)$.

Recall: $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \leq t \leq 1$, is a line segment between $\vec{r}_0, \vec{r}_1$.

$$\vec{r}(t) = (1-t) < -3,\ 2,\ -1 > +t < 0,\ -2,\ 3 >$$

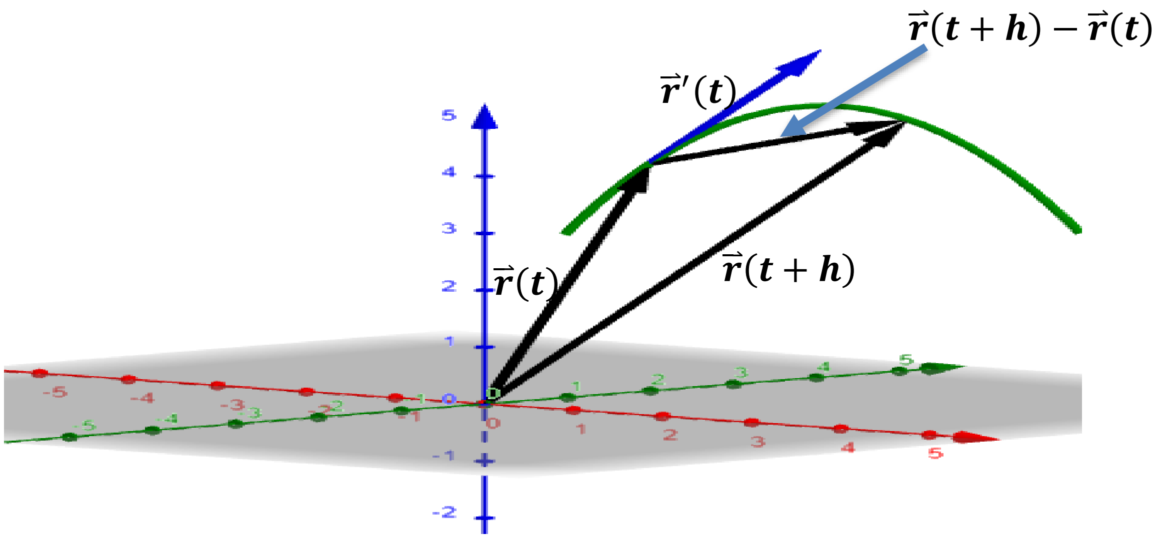
$$= < -3+3t,\ 2-2t,\ -1+t > + < 0,\ -2t,\ 3t >$$

$$\vec{r}(t) = < -3+3t,\ 2-4t,\ -1+4t >; \qquad 0 \leq t \leq 1$$

The corresponding parametric equations are:

$$x = -3+3t$$
$$y = 2-4t \qquad\qquad 0 \leq t \leq 1$$
$$z = -1+4t.$$

Def. $\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$ if the limit exists.



$\vec{r}\,'(t)$ is called the **tangent vector** for $\vec{r}(t)$, provided $\vec{r}\,'(t)$ exists and $\vec{r}\,'(t) \neq \vec{0}$ (i.e. the curve is "**regular**").

Def. The **tangent line to the curve, $\boldsymbol{C}$, at point, $\boldsymbol{P}$**, is defined to be the line through P parallel to the tangent vector $\vec{r}'(t)$.

Def. The **unit tangent vector** is: $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

Theorem: If $\vec{r}(t) = < f(t), g(t), m(t) > = f(t)\vec{\imath} + g(t)\vec{\jmath} + m(t)\vec{k}$,
where $f(t), g(t)$, and $m(t)$ are differentiable, then:

$$\vec{r}'(t) = < f'(t), g'(t), m'(t) > = f'(t)\vec{\imath} + g'(t)\vec{\jmath} + m'(t)\vec{k}.$$

Proof:

$$\vec{r}'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= < \lim_{h\to 0}\frac{f(t+h) - f(t)}{h}, \lim_{h\to 0}\frac{g(t+h) - g(t)}{h}, \lim_{h\to 0}\frac{m(t+h) - m(t)}{h} >$$

$$= < f'(t), g'(t), m'(t) >.$$

Ex. Find the derivative of $\vec{r}(t) = < t^3, \ln t\,, e^{-t} >$ and the unit tangent vector at $t = 1$.

$$\vec{r}'(t) = < 3t^2, \tfrac{1}{t}, -e^{-t} >$$

$$\vec{r}'(1) = < 3, 1, -e^{-1} >$$

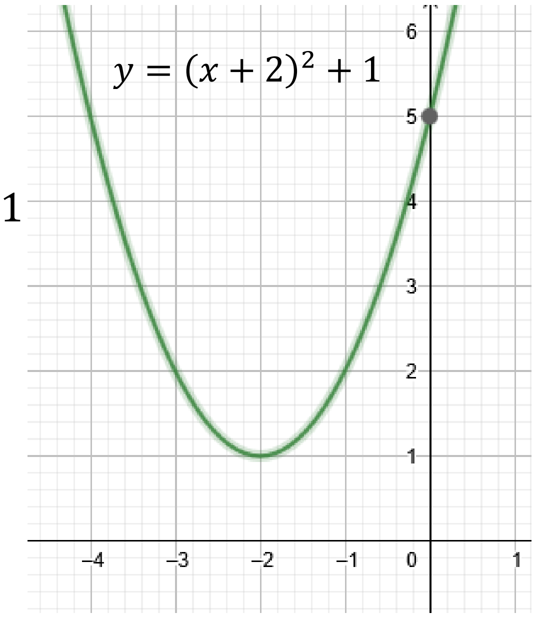
$$\vec{T}(1) = \frac{\vec{r}'(1)}{|\vec{r}'(1)|} = \frac{<3,1,-e^{-1}>}{\sqrt{3^2+1+e^{-2}}} = \frac{<3,1,-e^{-1}>}{\sqrt{10+e^{-2}}}$$

$$= < \frac{3}{\sqrt{10+e^{-2}}}, \frac{1}{\sqrt{10+e^{-2}}}, \frac{-e^{-1}}{\sqrt{10+e^{-2}}} >.$$

Ex. For the curve $\vec{r}(t) = < t-2, t^2+1 >$, sketch the curve and the tangent vector at $t=-1$.

$$x = t-2 \Rightarrow x+2 = t$$

$$y = t^2+1 \Rightarrow y = (x+2)^2+1$$

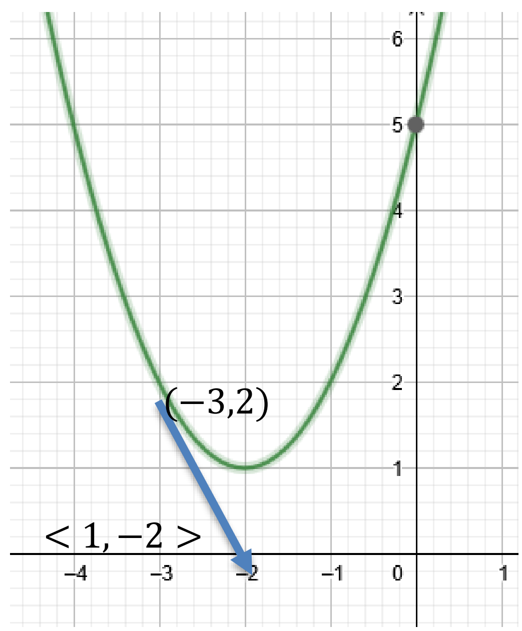


At $t=-1$; $x = -1-2 = -3$

$$y = (-1)^2+1 = 2$$

$$\vec{r}'(t) = < 1, 2t >$$

$$\vec{r}'(-1) = < 1, -2 >.$$



Ex. Find parametric equations for the tangent line to:

$x = t^2 + t$
$y = t^2 - t$ at $(2, 0, 2)$.
$z = t + 1$

$(x, y, z) = (t^2 + t, \ \ t^2 - t, \ \ t + 1) = (2,0,2)$

So $t + 1 = 2$ and $t = 1$.

We need to find the tangent line to this curve at $t = 1$.

$\vec{r}(t) = < t^2 + t, \ \ t^2 - t, \ \ t + 1 >$

$\vec{r}'(t) = < 2t + 1, \ \ 2t - 1, \ \ 1 >$.

At $(2, 0, 2)$ we have:

$$\vec{r}'(1) = < 3, 1, 1 >.$$

So we have a point on the line, $(2, 0, 2)$, and a direction vector, $\vec{v} = \vec{r}'(1) = < 3, 1, 1 >$.

Equation of tangent line:

$x = 2 + 3t$
$y = t$
$z = 2 + t$.

Ex. Suppose a particle following the path $\vec{c}(t) =< 4\cos(\pi t)\,,\ sin(\pi t),\ t^2 >$ flies off along the tangent line at $t_0 = 1$. Compute the position at $t_1 = 3$.

At $t_0 = 1$ the position of the particle is

$$\vec{c}(1) =< 4cos\pi,\ sin\pi,\ 1^2 >=< -4{,}0{,}1 >.$$

The tangent vector at $t_0 = 1$ is given by:

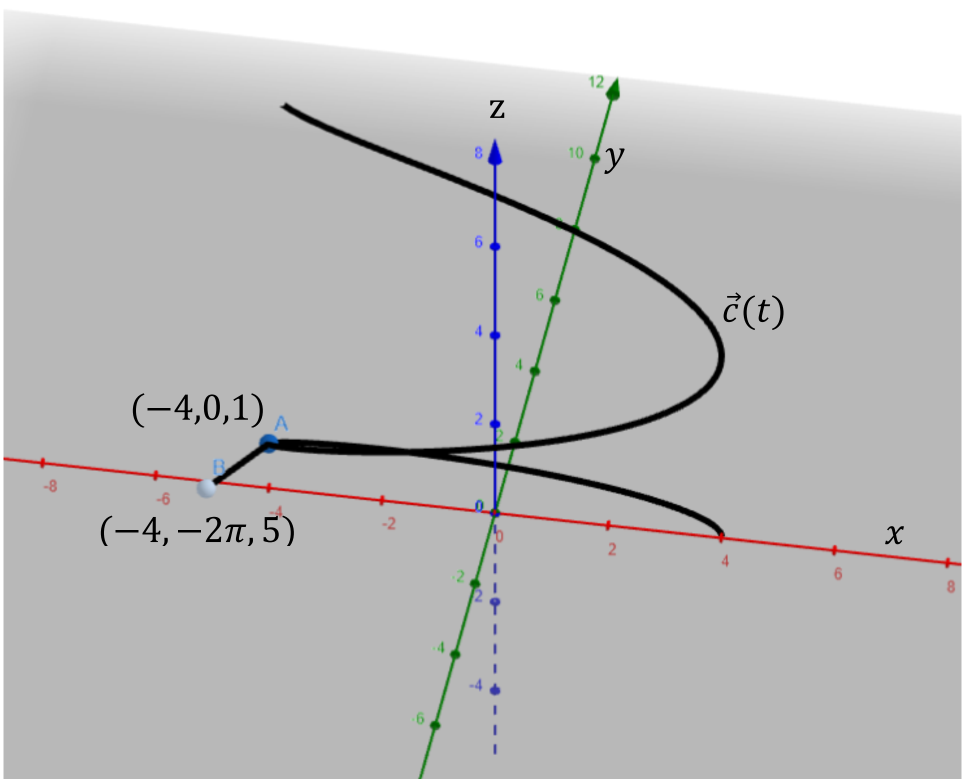
$$\vec{c}'(t) =< -4\pi\sin(\pi t)\,, \pi\cos(\pi t),\ 2t >$$

$$\vec{c}'(1) =< 0, -\pi,\ 2 >.$$

Thus the vector equation of the tangent line at $t = 1$ is:

$$\vec{l}(s) = \vec{c}(1) + s\left(\vec{c}'(1)\right) =< -4{,}0{,}1 > +s < 0, -\pi,\ 2 >.$$

$$=< -4,\ -s\pi,\ 2s + 1\ >.$$



Notice that at $t = 1,\ s = 0$. Thus to find the position of the particle at $t = 3$ we need to substitute $s = 2$ into the equation of the tangent line.

Position of particle at $t = 3$, i.e., when $s = 2$:

$$\vec{l}(2) =< -4, -2\pi,\ 5 >.$$

Theorem: If $\vec{u}(t), \vec{v}(t)$ different vector functions, c a constant, and $f(t)$ is a differentiable real valued function, then:

1. $\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt} = \vec{u}'(t) + \vec{v}'(t)$

2. $\frac{d}{dt}\left(c\vec{u}(t)\right) = c\,\frac{d\vec{u}}{dt} = c\vec{u}'(t)$

3. $\frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

4. $\frac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

5. $\frac{d}{dt}\left(\vec{u}(t) \times \vec{v}(t)\right) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

6. $\frac{d}{dt}\left(\vec{u}\big(f(t)\big)\right) = f'(t)\,\vec{u}'\big(f(t)\big)$ chain rule.

Proof of #4:

$$\vec{u}(t) = < f_1(t), f_2(t), f_3(t) > ; \quad \vec{v}(t) = < g_1(t), g_2(t), g_3(t) >$$

$$\vec{u}(t) \cdot \vec{v}(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)$$

$$\frac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = f_1(t)g'_1(t) + f'_1(t)g_1(t) + f'_2(t)g_2(t) + f_2(t)g'_2(t) + f'_3(t)g_3(t) + f_3(t)g'_3(t)$$

$$= \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t).$$