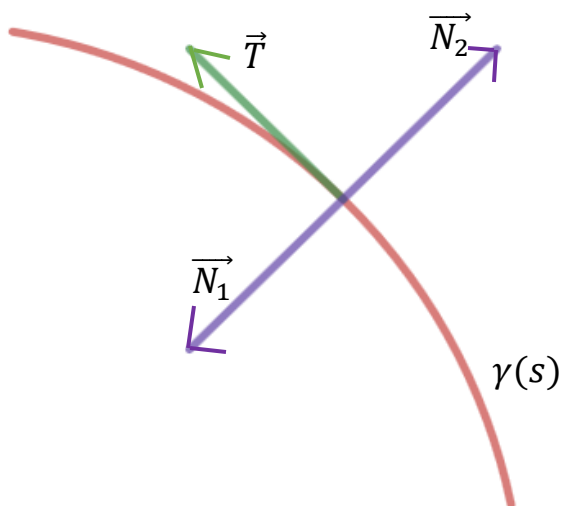


Plane Curves

For plane curves it is possible to define curvature so that it can be positive or negative.

Suppose $\gamma(s)$ is a unit speed parameterization of a plane curve γ . Then $\gamma'(s)$ is a unit tangent vector to γ at $\gamma(s)$. Let's call this tangent vector, $\vec{T} = \gamma'(s)$.

Since γ is a plane curve there are two unit vectors perpendicular to \vec{T} .



We will choose the signed unit normal, \vec{N}_s (\vec{N}_1 above), of γ to be the unit vector obtained by rotating \vec{T} counterclockwise by $\frac{\pi}{2}$.

Note: If $\vec{T} = (a, b)$ then $\vec{N}_s = (-b, a)$.

Since $\gamma'(s) \cdot \gamma'(s) = 1$, by differentiating this equation we get:

$$\gamma'(s) \cdot \gamma''(s) + \gamma''(s) \cdot \gamma'(s) = 0$$

or

$$\gamma' \cdot \gamma'' = 0.$$

Thus γ'' is perpendicular to $\vec{T} = \gamma'(s)$, just as \vec{N}_s is. Thus we can write:

$$\gamma''(s) = \kappa_s \vec{N}_s$$

κ_s is called the **signed curvature** of γ .

Notice that since $\|\vec{N}_s\| = 1$ we have:

$$\kappa = \|\gamma''(s)\| = |\kappa_s| \|\vec{N}_s\| = |\kappa_s|$$

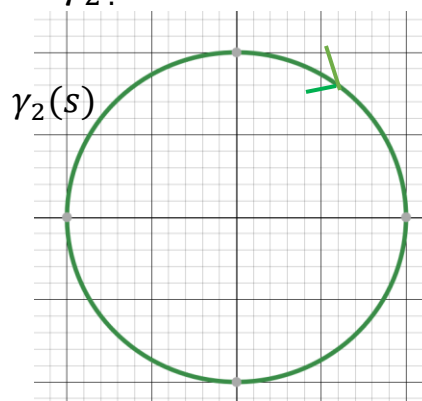
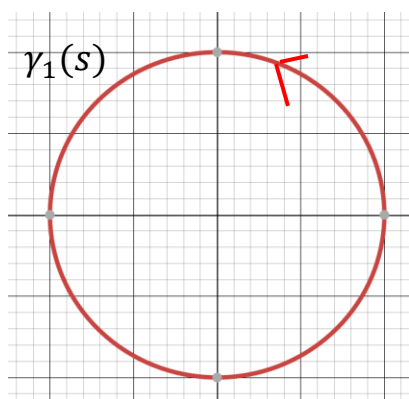
where κ is the (unsigned) curvature of γ .

Ex. Let's consider two unit speed parameterizations of the unit circle, one going counterclockwise as s increases, and one going clockwise as s increases

$$\gamma_1(s) = (\cos(s), \sin(s))$$

$$\gamma_2(s) = (\cos(s), -\sin(s)).$$

Calculate the signed curvatures of γ_1 and γ_2 .



$$\vec{T}_1 = \gamma'_1(s) = (-\sin(s), \cos(s))$$

$$\vec{N}_s(s) = (-\cos(s), -\sin(s))$$

$$\gamma''_1(s) = (-\cos(s), -\sin(s)) = 1(\vec{N}_s)$$

So the signed curvature of γ_1 is equal to 1 at all points.

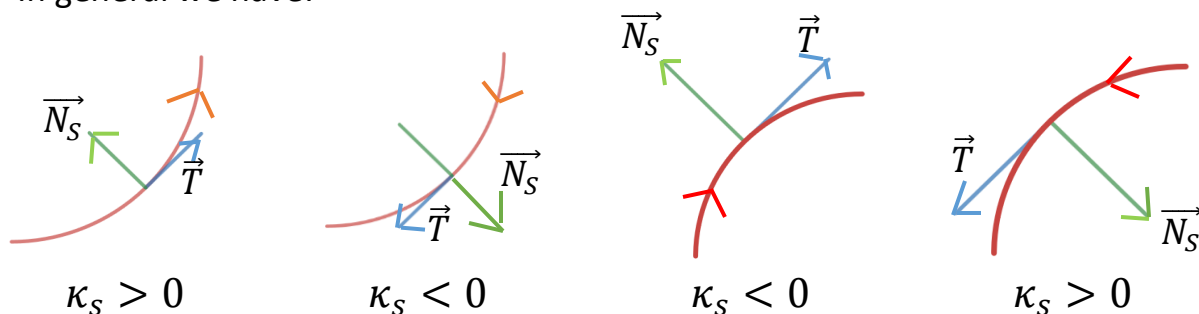
$$\vec{T}_2 = \gamma'_2(s) = (-\sin(s), -\cos(s))$$

$$\vec{N}_s(s) = (\cos(s), -\sin(s))$$

$$\gamma''_2(s) = (-\cos(s), \sin(s)) = -1(\vec{N}_s)$$

So the signed curvature of γ_2 is equal to -1 at all points.

In general we have:



If $\gamma(t)$ is a regular plane curve (not necessarily unit speed) we define its unit tangent \vec{T} , its signed normal \vec{N}_s , and its signed curvature κ_s to be those of a unit speed parametrization of γ . Thus we have:

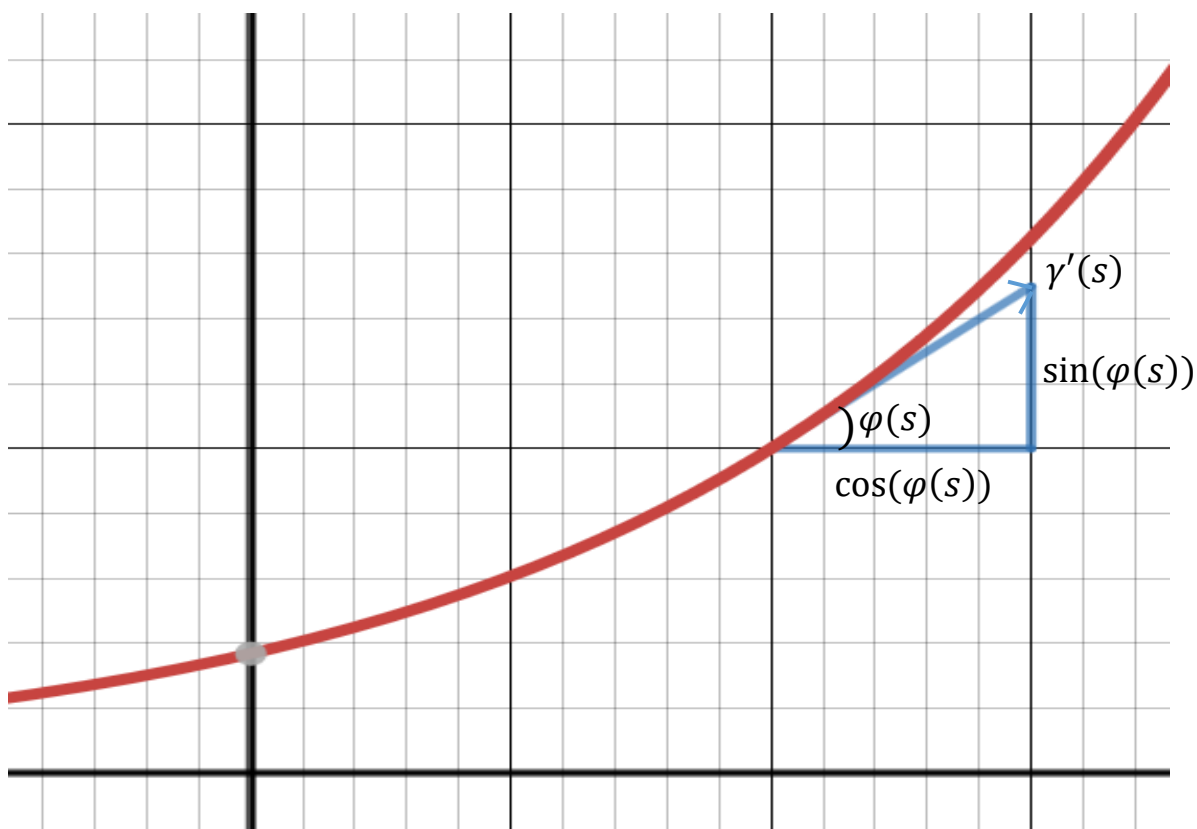
$$\vec{T} = \frac{d\gamma}{ds} = \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}} = \frac{\gamma'(t)}{\|\gamma'(t)\|}.$$

\vec{N}_s is again obtained by rotating \vec{T} by $\frac{\pi}{2}$ counterclockwise and

$$\frac{d}{dt}(\vec{T}) = \frac{d(\vec{T})}{ds} \frac{ds}{dt} = \kappa_s \frac{ds}{dt} \vec{N}_s = \kappa_s \|\gamma'(t)\| \vec{N}_s.$$

The signed curvature has a simple geometric interpretation in terms of the rate at which the tangent vector rotates. Let γ be a unit speed curve, then if $\varphi(s)$ is the angle the tangent vector makes with the x -axis we have:

$$\gamma'(s) = (\cos(\varphi(s)), \sin(\varphi(s)))$$



$\varphi(s)$ is called the **turning angle** of γ .

Proposition: Let $\gamma(s)$ be a unit speed plane curve, then $\kappa_s = \frac{d\varphi}{ds}$.

Proof: $\vec{T} = (\cos \varphi, \sin \varphi)$

$$\begin{aligned} \frac{d\vec{T}}{ds} &= \left(-(\sin \varphi) \frac{d\varphi}{ds}, (\cos \varphi) \frac{d\varphi}{ds} \right) \\ &= \frac{d\varphi}{ds} (-\sin \varphi, \cos \varphi) = \frac{d\varphi}{ds} \vec{N}_s \\ &\Rightarrow \kappa_s = \frac{d\varphi}{ds}. \end{aligned}$$

Now we can derive a formula to κ_s for any smooth, regular curve in a plane.

$$\kappa_s = \frac{d\varphi}{ds} = \frac{\frac{d\varphi}{dt}}{\frac{ds}{dt}}$$

Suppose $\gamma(t) = (x(t), y(t))$, then:

$$\frac{ds}{dt} = \sqrt{(x')^2 + (y')^2}, \text{ since } \frac{ds}{dt} = \|\gamma'(t)\|.$$

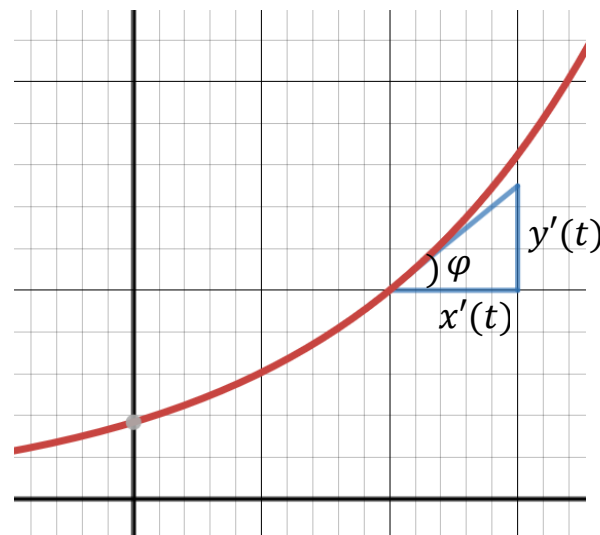
Since $\gamma'(t) = (x'(t), y'(t))$ is tangent to $\gamma(t)$

$$\tan \varphi = \frac{y'(t)}{x'(t)} \text{ or } \varphi = \tan^{-1}\left(\frac{y'}{x'}\right)$$

$$\frac{d\varphi}{dt} = \frac{1}{1 + \left(\frac{y'}{x'}\right)^2} \left(\frac{x'y'' - y'x''}{(x')^2} \right)$$

$$\frac{d\varphi}{dt} = \frac{1}{(x')^2 + (y')^2} (x'y'' - y'x'')$$

$$\kappa_s = \frac{d\varphi}{ds} = \frac{\frac{d\varphi}{dt}}{\frac{ds}{dt}} = \frac{x'y'' - y'x''}{((x')^2 + (y')^2)^{\frac{3}{2}}}.$$



Ex. Find the signed curvature of $\gamma(t) = (\cos t + t \sin t, \sin t - t \cos t)$.

$$x(t) = \cos t + t \sin t$$

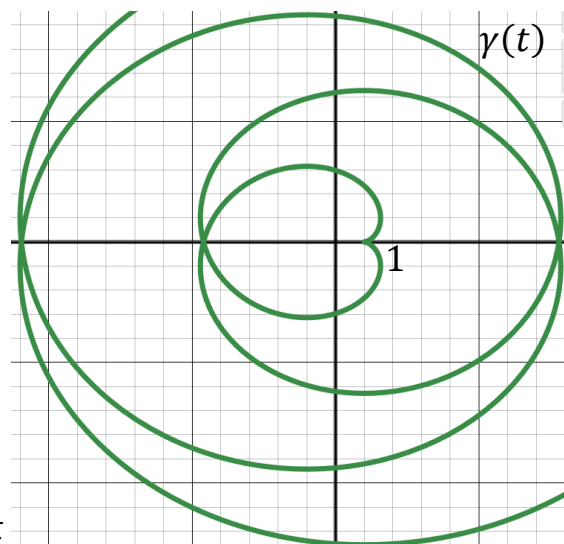
$$\begin{aligned} x'(t) &= -\sin t + t \cos t + \sin t \\ &= t \cos t \end{aligned}$$

$$x''(t) = -t \sin t + \cos t$$

$$y(t) = \sin t - t \cos t$$

$$y'(t) = \cos t + t \sin t - \cos t = t \sin t$$

$$y''(t) = t \cos t + \sin t$$



$$\begin{aligned} \kappa_s &= \frac{x'y'' - y'x''}{((x')^2 + (y')^2)^{\frac{3}{2}}} \\ &= \frac{(t \cos t)(t \cos t + \sin t) - (t \sin t)(-t \sin t + \cos t)}{(t^2 \cos^2 t + t^2 \sin^2 t)^{\frac{3}{2}}} = \frac{t^2}{|t|^3} = \frac{1}{|t|}. \end{aligned}$$

Ex. Suppose γ is a curve in \mathbb{R}^2 . Using the formula for the curvature, κ , of a curve in \mathbb{R}^3 (or \mathbb{R}^2), and the formula for the signed curvature, κ_s , in \mathbb{R}^2 , show $|\kappa_s| = \kappa$.

$$\text{Let } \gamma(t) = (x(t), y(t), 0).$$

$$\gamma'(t) = (x'(t), y'(t), 0).$$

$$\gamma''(t) = (x''(t), y''(t), 0).$$

$$\kappa = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3}$$

$$\gamma'' \times \gamma' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'' & y'' & 0 \\ x' & y' & 0 \end{vmatrix} = (x''y' - y'x'')\vec{k}$$

$$\|\gamma'' \times \gamma'\| = |x''y' - y'x''|$$

$$\|\gamma'\|^3 = (\sqrt{(x')^2 + (y')^2})^3 = ((x')^2 + (y')^2)^{\frac{3}{2}}$$

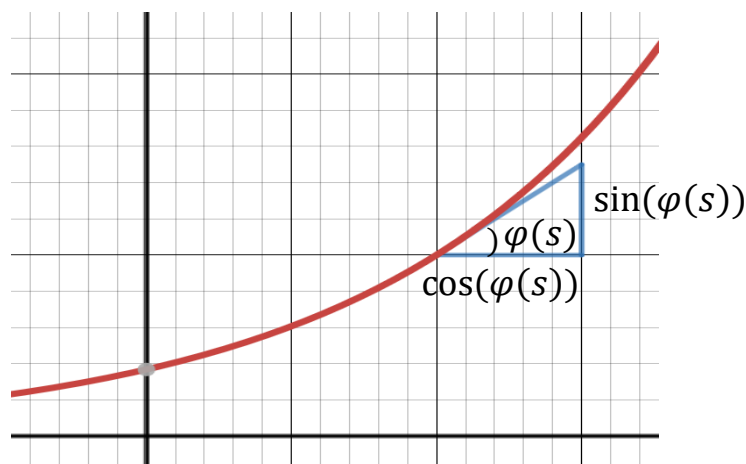
$$\kappa = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3} = \frac{|x''y' - y'x''|}{((x')^2 + (y')^2)^{\frac{3}{2}}} = |\kappa_s|$$

The fact that $\kappa_s = \frac{d\varphi}{ds}$ has an interesting consequence for the total curvature of unit speed closed curves in a plane. If we let l be the length of the closed curve, then:

$$\text{Total signed curvature} = \int_0^l \frac{d\varphi}{ds} ds = \varphi(l) - \varphi(0) = 2\pi n; \quad n \in \mathbb{Z}.$$

Fundamental Theorem of Plane Curves: let $\kappa: (\alpha, \beta) \rightarrow \mathbb{R}$ be any smooth function. Then, there is a unit speed curve $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^2$ whose signed curvature is κ . Furthermore, if $\bar{\gamma}: (\alpha, \beta) \rightarrow \mathbb{R}^2$ is any other unit speed curve whose signed curvature is κ , then γ and $\bar{\gamma}$ differ by a rotation and/or a translation.

Idea of Proof: Given any smooth function $\kappa: (\alpha, \beta) \rightarrow \mathbb{R}$ we want to construct a curve $\gamma(s)$ such that $\kappa_s = \frac{d\varphi}{ds} = \kappa$ for $\gamma(s)$.



$$\gamma'(s) = \vec{T} = (\cos \varphi, \sin \varphi)$$

$$\gamma''(s) = \frac{d\vec{T}}{ds} = \left(-(\sin \varphi) \frac{d\varphi}{ds}, (\cos \varphi) \frac{d\varphi}{ds} \right).$$

We can find a curve $\gamma(s)$ with $\frac{d\varphi}{ds} = \kappa$ by integrating this last expression twice:

$$\gamma'(s) = \left(\int -(\sin \varphi) \frac{d\varphi}{ds} ds, \int (\cos \varphi) \frac{d\varphi}{ds} ds \right)$$

$$= (\cos \varphi(s) + C_1, \sin \varphi(s) + C_2)$$

$$\gamma(s) = \left(\int (\cos \varphi(s) + C_1) ds, \int (\sin \varphi(s) + C_2) ds \right).$$