Principal Curvatures of a Surface

A linear transformation, $P: V \to V$, is called self adjoint if

$$< P\vec{w}_1, \vec{w}_2 > = < \vec{w}_1, P\vec{w}_2 >$$

where $< , >$ is an inner product of vectors. In the case of the Weingarten map, $P = W = -D_p\widetilde{G}$, and vectors $\vec{w}_1 = (a_1, b_1)$ and $\vec{w}_2 = (a_2, b_2)$. Let's show that W is self adjoint.

Let $\vec{w}_1, \vec{w}_2 \in T_pS$ and write:

$$\vec{w}_1 = a_1\vec{\Phi}_u + b_1\vec{\Phi}_v$$

$$\vec{w}_2 = a_2\vec{\Phi}_u + b_2\vec{\Phi}_v.$$

Then:

$$< W\vec{w}_1, \vec{w}_2 > = \left(W\left(a_1\vec{\Phi}_u + b_1\vec{\Phi}_v\right)\right) \cdot \left(a_2\vec{\Phi}_u + b_2\vec{\Phi}_v\right)$$

$$= \left(-a_1\vec{N}_u - b_1\vec{N}_v\right) \cdot \left(a_2\vec{\Phi}_u + b_2\vec{\Phi}_v\right)$$

since $W\left(\vec{\Phi}_u\right) = -\vec{N}_u,\ W\left(\vec{\Phi}_v\right) = -\vec{N}_v$.

Now using the identity: $\vec{N}_u \cdot \vec{\Phi}_v = \vec{\Phi}_u \cdot \vec{N}_v$ we get

$$< W\vec{w}_1, \vec{w}_2 > = \left(a_1\vec{\Phi}_u + b_1\vec{\Phi}_v\right) \cdot \left(-a_2\vec{N}_u - b_2\vec{N}_v\right)$$

$$= < \vec{w}_1, W(\vec{w}_2) >$$

So W is self adjoint.

There is a theorem from linear algebra that says that if $P: V \to V$ is a self adjoint linear map of V, a two (or n) dimensional vector space into itself, then there exists an orthonormal basis $\vec{e}_1$ and $\vec{e}_2$ of V such that $P(\vec{e}_1) = \lambda_1 \vec{e}_1$ and $P(\vec{e}_2) = \lambda_2 \vec{e}_2$ (that is, $\vec{e}_1$ and $\vec{e}_2$ are eigenvectors and λ_1, λ_2 are eigenvalues of P).

In the case of the Weingarten map: $W: T_pS \to T_pS$, there are orthonormal tangent vectors to S, $\vec{t}_1, \vec{t}_2 \in T_pS$ such that:

$$W(\vec{t}_1) = \kappa_1 \vec{t}_1$$

$$W(\vec{t}_2) = \kappa_2 \vec{t}_2.$$

Then with respect to the basis vectors $\vec{t}_1$ and $\vec{t}_2$, the matrix W is given by:

$$W = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

If $\kappa_1 = \kappa_2 = \kappa$, then:

$$W = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix}$$ (i.e. a multiple of the identity matrix).

Def. κ_1, κ_2 are called the **principal curvatures** of S and $\vec{t}_1, \vec{t}_2$ are called **principal vectors** corresponding to κ_1, κ_2.

Def. If $\kappa_1 = \kappa_2$ for $p \in S$, we say p is an **umbilic point** of S.

Prop. If κ_1 and κ_2 are the principal curvatures of $p \in S$, then the mean and Gaussian curvatures are given by:

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)\,, \quad K = (\kappa_1)(\kappa_2).$$

Proof: The determinant and trace of a linear transformation do not change when the basis is changed. Thus, using the principal vectors as a basis, the Weingarten map is represented by:

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Hence:

$K = \det(W) = (\kappa_1)(\kappa_2)$ and

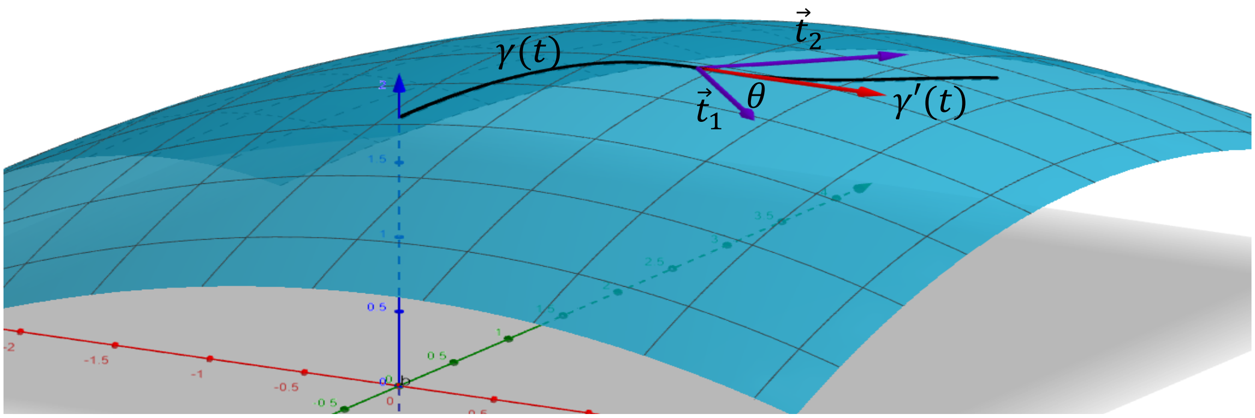
$H = \frac{1}{2}\mathrm{trace}(W) = \frac{1}{2}(\,\kappa_1 + \kappa_2).$

One application of principal curvatures and principal vectors is given by:

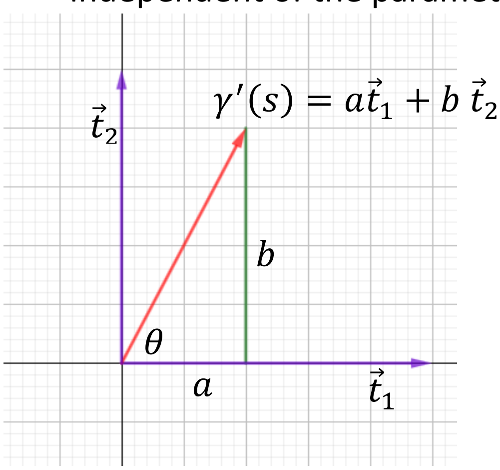
Theorem (Euler's Theorem): Let γ be a curve on an oriented surface, S, and let κ_1, κ_2 be the principal curvatures of $\overrightarrow{\Phi}$, with non-zero principal vectors, $\vec{t}_1$ and $\vec{t}_2$. Then the normal curvature of γ is:

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

where θ is the angle between $\vec{t}_1$ and $\gamma'(t)$.



Proof: Let $\vec{t}_1$ and $\vec{t}_2$ be an orthonormal basis for T_pS with $W(\vec{t}_1) = \kappa_1 \vec{t}_1$ and $W(\vec{t}_2) = \kappa_2 \vec{t}_2$. We can take γ to be unit speed since curvatures are independent of the parametrization, so we can write:



$\gamma'(s) = a\vec{t}_1 + b\,\vec{t}_2$; where

$$cos\theta = \frac{a}{\|\gamma'(s)\|} = a$$

$$sin\theta = \frac{b}{\|\gamma'(s)\|} = b; \quad \text{so,}$$

$$\gamma'(s) = (\cos\theta)\vec{t}_1 + (\sin\theta)\vec{t}_2.$$

We saw earlier that:

$$\kappa_n = < W(\gamma'(s)), \gamma'(s) >$$

$$= < W\left((\cos\theta)\vec{t}_1 + (\sin\theta)\vec{t}_2\right), (\cos\theta)\vec{t}_1 + (\sin\theta)\vec{t}_2 >$$

$$= < (\cos\theta)W(\vec{t}_1) + (\sin\theta)W(\vec{t}_2), (\cos\theta)\vec{t}_1 + (\sin\theta)\vec{t}_2 >$$

$$= < (\cos\theta)\kappa_1(\vec{t}_1) + (\sin\theta)\kappa_2(\vec{t}_2), (\cos\theta)\vec{t}_1 + (\sin\theta)\vec{t}_2 >$$

$$\kappa_n = (\cos^2\theta)\kappa_1 + (\sin^2\theta)\kappa_2.$$

Cor: The principal curvatures at a point on a surface are the maximum and minimum values of the normal curvature of all curves on the surface passing through the point. Moreover, the principal vectors are the tangent vectors of the curves giving these maximum and minimum values.

Proof: If $\kappa_1 \neq \kappa_2$, assume $\kappa_1 > \kappa_2$, then we can write:

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$
$$= \kappa_1(1 - \sin^2\theta) + \kappa_2 \sin^2\theta$$
$$\kappa_n = \kappa_1 - (\kappa_1 - \kappa_2)\sin^2\theta.$$

Thus, $\kappa_n \leq \kappa_1$ since $\kappa_1 - \kappa_2 > 0$ and $\sin^2\theta \geq 0$.

In addition, $\kappa_n = \kappa_1$ if, and only if, $\theta = 0$ or π, i.e., $\gamma'(s)$ is parallel to $\vec{t}_1$.

Similarly, we know:

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2(1 - \cos^2\theta) = \kappa_2 + (\kappa_1 - \kappa_2)\cos^2\theta$$

Thus, $\kappa_n \geq \kappa_2$ with equality if, and only if, $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ i.e., $\gamma'(s)$ is parallel to $\vec{t}_2$.

If $\kappa_1 = \kappa_2$, then $\kappa_n = \kappa_1 \cos^2\theta + \kappa_1 \sin^2\theta = \kappa_1$ and every unit tangent vector to the surface is a principal vector since every unit vector is an eigenvector:

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \kappa_1 \begin{pmatrix} a \\ b \end{pmatrix}.$$

Cor. The mean curvature, H, is the average value of the normal curvature over $0 \le \theta \le 2\pi$.

Proof: $\frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} \kappa_n d\theta = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi}[(\cos^2\theta)\kappa_1 + (\sin^2\theta)\kappa_2]d\theta$

$$= \frac{1}{2\pi}[\kappa_1 \int_{\theta=0}^{2\pi} \cos^2\theta d\theta + \kappa_2 \int_{\theta=0}^{2\pi} \sin^2\theta d\theta]$$

$$= \frac{1}{2\pi}[\kappa_1 \int_{\theta=0}^{2\pi} \left(\frac{1}{2} + \frac{1}{2}cos2\theta\right) d\theta + \kappa_2 \int_{\theta=0}^{2\pi} \left(\frac{1}{2} - \frac{1}{2}cos2\theta\right) d\theta]$$

$$= \frac{1}{2\pi}[\kappa_1(\pi) + \kappa_2(\pi)] = \frac{1}{2}[\kappa_1 + \kappa_2] = H.$$

Recall from linear algebra, if P is a linear transformation of a vector space V into V, then $\vec{w} \in V$ is an **eigenvector** of P if $P(\vec{w}) = \lambda\vec{w}$. In this case, λ is called the **eigenvalue** of P. This is equivalent to saying:

$$(P - \lambda I)\vec{w} = \vec{0}.$$

To find the eigenvalues we solve the equation:

$\det(P - \lambda I) = 0$ for λ.

To find the eigenvectors, $\vec{w}$, we let $\vec{w} = a\vec{w}_1 + b\vec{w}_2$; where $\vec{w}_1$ and $\vec{w}_2$ are basis vectors for V and solve for a, b in:

$$(P - \lambda I)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Given a surface patch $\overrightarrow{\Phi}(u,v)$ for a surface, S, how do we calculate κ_1 and κ_2, the principal curvatures?

κ_1 and κ_2 are the eigenvalues of W, the Weingarten map, and we already saw that:

$$W = (F_1^{-1})(F_2)$$

where F_1 and F_2 are the matrix representations of the first and second fundamental forms with respect to the basis $\overrightarrow{\Phi}_u$ and $\overrightarrow{\Phi}_v$.

Thus we need to solve: $\det(F_1^{-1}F_2 - \kappa I) = 0$ for κ.

Notice that:
$$\begin{aligned}\det(F_1^{-1}F_2 - \kappa I) &= \det(F_1^{-1}(F_2 - \kappa F_1)) \\ &= \det(F_1^{-1})\det(F_2 - \kappa F_1) = 0.\end{aligned}$$

Since $\det(F_1^{-1}) \neq 0$, this is equivalent to solving: $\det(F_2 - \kappa F_1) = 0.$

We can find the principal curvature vectors, $\vec{t}_1$ and $\vec{t}_2$, which are the eigenvectors of $W = F_1^{-1}F_2$ by letting $\vec{t} = a\overrightarrow{\Phi}_u + b\overrightarrow{\Phi}_v$ and solving:

$$(F_1^{-1}F_2 - \kappa I)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is the equivalent to solving:

$$F_1^{-1}(F_2 - \kappa F_1)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$(F_2 - \kappa F_1)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Recall that with respect to the basis of T_pS given by $\overrightarrow{\Phi}_u$ and $\overrightarrow{\Phi}_v$, we have:

$$F_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

where E, F, G, L, M, N are defined as before.

Proposition: The principal curvatures are the roots of:

$$\begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0$$

and the principal curvature vectors to the principal curvature, κ, are the unit tangent vectors, $\vec{t} = a\overrightarrow{\Phi}_u + b\overrightarrow{\Phi}_v$ such that:

$$\begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Ex. Find the principal curvatures and principal curvature vectors for the sphere of radius, R.

We saw earlier that for the parametrization of S^2 by:

$$\overrightarrow{\Phi}(\phi, \theta) = (\mathrm{Rcos}\,\theta \sin\phi\,, \mathrm{Rsin}\,\theta \sin\phi\,, \mathrm{Rcos}\,\phi)$$

We have:

$$F_1 = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\phi \end{pmatrix}$$

and

$$F_2 = \begin{pmatrix} -R & 0 \\ 0 & -R\sin^2\phi \end{pmatrix}.$$

That is:

$$E = R^2, \qquad F = 0, \qquad G = R^2 \sin^2 \phi,$$
$$L = -R, \qquad M = 0, \qquad N = -R \sin^2 \phi.$$

Thus, to find the eigenvalues of W we solve $\det(F_2 - \kappa F_1) = 0$:

$$\begin{vmatrix} -R - \kappa R^2 & 0 \\ 0 & -R \sin^2 \phi - \kappa R^2 \sin^2 \phi \end{vmatrix} = 0$$

$$\begin{vmatrix} -R(1 + \kappa R) & 0 \\ 0 & -R\,(\sin^2 \phi)\,(1 + \kappa R) \end{vmatrix} = 0$$

$$R^2 \sin^2 \phi\,(1 + \kappa R)^2 = 0.$$

$\kappa = -\frac{1}{R}$, a double root so:

$$\kappa_1 = \kappa_2 = -\frac{1}{R}.$$

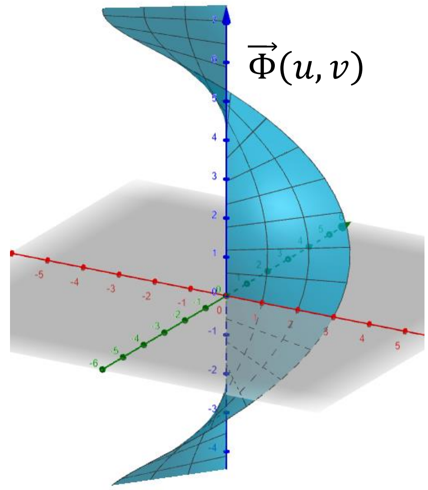
Since $\kappa_1 = \kappa_2$, every unit vector $\vec{t} \in T_p S$ is a principal curvature vector (for all points $p \in S$). Hence, every point of S is an umbilic point.

Note: κ_1, κ_2 depend on which direction we choose for $\vec{N}$, the unit normal to S. If we had chosen $-\vec{N}$ for S^2 we would have gotten:

$$\kappa_1 = \kappa_2 = \frac{1}{R}.$$

Ex. Find the principal curvatures and principal curvature vectors for the helicoid given by:

$$\vec{\Phi}(u,v) = (v\cos u\,, v\sin u\,, 2u); \quad 0 \le v \le 3, u \in \mathbb{R}.$$



$$\vec{\Phi}_u = (-v\sin u, v\cos u\,, 2)$$

$$\vec{\Phi}_v = (\cos u, \sin u\,, 0)$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -v\sin u & v\cos u & 2 \\ \cos u & \sin u & 0 \end{vmatrix}$$

$$= -2\sin u\,\vec{\imath} + 2\cos u\,\vec{\jmath} - v\vec{k}$$

$$\left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\| = \sqrt{4\sin^2 u + 4\cos^2 u + v^2} = \sqrt{4+v^2}$$

$$\vec{N} = \frac{\vec{\Phi}_u \times \vec{\Phi}_v}{\left\|\vec{\Phi}_u \times \vec{\Phi}_v\right\|} = \frac{(-2\sin u, 2\cos u, -v)}{\sqrt{4+v^2}}$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = (-v\sin u, v\cos u\,, 2) \cdot (-v\sin u, v\cos u\,, 2) = v^2 + 4$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = (-v\sin u, v\cos u\,, 2) \cdot (\cos u, \sin u\,, 0) = 0$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = (\cos u, \sin u\,, 0) \cdot (\cos u, \sin u\,, 0) = 1.$$

$$\vec{\Phi}_{uu} = (-v\cos u\,, -v\sin u\,, 0)$$
$$\vec{\Phi}_{uv} = (-\sin u\,, \cos u\,, 0)$$
$$\vec{\Phi}_{vv} = (0,0,0)$$

$$L = \vec{\Phi}_{uu} \cdot \vec{N} = (-v\cos u\,, -v\sin u\,, 0) \cdot \frac{(-2\sin u, 2\cos u, -v)}{\sqrt{4+v^2}} = 0$$

$$M = \vec{\Phi}_{uv} \cdot \vec{N} = (-\sin u\,, \cos u\,, 0) \cdot \frac{(-2\sin u, 2\cos u, -v)}{\sqrt{4+v^2}} = \frac{2}{\sqrt{4+v^2}}$$

$$N = \vec{\Phi}_{vv} \cdot \vec{N} = (0,0,0) \cdot \frac{(-2\sin u, 2\cos u, -v)}{\sqrt{4+v^2}} = 0.$$

So we have:

$$F_1 = \begin{pmatrix} v^2+4 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} 0 & \frac{2}{\sqrt{4+v^2}} \\ \frac{2}{\sqrt{4+v^2}} & 0 \end{pmatrix}.$$

So to find the principal curvatures we must solve $\det(F_2 - \kappa F_1) = 0$:

$$\begin{vmatrix} -\kappa(v^2+4) & \frac{2}{\sqrt{4+v^2}} \\ \frac{2}{\sqrt{4+v^2}} & -\kappa \end{vmatrix} = 0$$

$$\kappa^2(v^2+4) - \frac{4}{4+v^2} = 0$$

$$\kappa^2 = \frac{4}{(4+v^2)^2}$$

$$\kappa = \pm\frac{2}{4+v^2}\,.$$

To find $\vec{t}_1$ and $\vec{t}_2$ we solve $(F_2 - \kappa F_1)\binom{a}{b} = \binom{0}{0}$:

$$\kappa = \frac{2}{4+v^2}; \qquad \vec{t}_1 = a\overrightarrow{\Phi}_u + b\overrightarrow{\Phi}_v.$$

$$\begin{pmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{pmatrix}\binom{a}{b} = \binom{0}{0}$$

$$\begin{pmatrix} -2 & \frac{2}{\sqrt{4+v^2}} \\ \frac{2}{\sqrt{4+v^2}} & -\frac{2}{4+v^2} \end{pmatrix}\binom{a}{b} = \binom{0}{0}$$

$$-2a + \frac{2}{\sqrt{4+v^2}}b = 0 \implies b = a\sqrt{4+v^2}.$$

So any vector of the form $\left(a, a\sqrt{4+v^2}\right)$ will get mapped to $(0,0)$. We want a unit vector so we must find a such that:

$$\begin{aligned} 1 = \vec{t}_1 \cdot \vec{t}_1 &= a(1, \sqrt{4+v^2}) \cdot a(1, \sqrt{4+v^2}) \\ &= a^2(\overrightarrow{\Phi}_u + \sqrt{4+v^2}\,\overrightarrow{\Phi}_v) \cdot (\overrightarrow{\Phi}_u + \sqrt{4+v^2}\,\overrightarrow{\Phi}_v) \\ &= a^2(E + 2\sqrt{4+v^2}F + (4+v^2)G) \\ &= a^2(4+v^2+4+v^2) = 2a^2(4+v^2). \end{aligned}$$

Solving for a we get:

$$a = \pm\frac{1}{\sqrt{2}\sqrt{4+v^2}}.$$

Thus $\vec{t}_1$ can be either:

$$\vec{t}_1 = \pm\frac{1}{\sqrt{2}\sqrt{4+v^2}}\left(1, \sqrt{4+v^2}\right).$$

Similarly, if $\kappa = -\frac{2}{4+v^2}$, then we solve:

$$\begin{pmatrix} 2 & \frac{2}{\sqrt{4+v^2}} \\ \frac{2}{\sqrt{4+v^2}} & \frac{2}{4+v^2} \end{pmatrix}\binom{a}{b} = \binom{0}{0}$$

$$2a + \frac{2}{\sqrt{4+v^2}}b = 0 \implies b = -a\sqrt{4+v^2}.$$

So any vector of the form $\left(a, -a\sqrt{4+v^2}\right)$ will get mapped to $(0,0)$.

Again we want to find an a so that this is a unit vector.

$$\begin{aligned} 1 = \vec{t}_2 \cdot \vec{t}_2 &= a(1, -\sqrt{4+v^2}) \cdot a(1, -\sqrt{4+v^2}) \\ &= a^2(\vec{\Phi}_u - \sqrt{4+v^2}\vec{\Phi}_v) \cdot (\vec{\Phi}_u - \sqrt{4+v^2}\vec{\Phi}_v) \\ &= a^2(E - 2\sqrt{4+v^2}F + (4+v^2)G) \\ &= a^2(4+v^2+4+v^2) = 2a^2(4+v^2). \end{aligned}$$

Solving for a we get:

$$a = \pm\frac{1}{\sqrt{2}\sqrt{4+v^2}}$$

Thus $\vec{t}_2$ can be either:

$$\vec{t}_2 = \pm\frac{1}{\sqrt{2}\sqrt{4+v^2}}\left(1, -\sqrt{4+v^2}\right).$$

Notice that $\vec{t}_1$ and $\vec{t}_2$ are perpendicular to eachother since if we take $\vec{t}_1 = \frac{1}{\sqrt{2}\sqrt{4+v^2}}\left(1, \sqrt{4+v^2}\right)$ and $\vec{t}_2 = \frac{1}{\sqrt{2}\sqrt{4+v^2}}\left(1, -\sqrt{4+v^2}\right)$ for example:

$$\vec{t}_1 \cdot \vec{t}_2 = \frac{1}{\sqrt{2}\sqrt{4+v^2}}\left(1, \sqrt{4+v^2}\right) \cdot \frac{1}{\sqrt{2}\sqrt{4+v^2}}\left(1, -\sqrt{4+v^2}\right)$$

$$= \frac{1}{2(4+v^2)}\left(\vec{\Phi}_u + \sqrt{4+v^2}\,\vec{\Phi}_v\right) \cdot \left(\vec{\Phi}_u - \sqrt{4+v^2}\,\vec{\Phi}_v\right)$$

$$= \frac{1}{2(4+v^2)}\left(\vec{\Phi}_u \cdot \vec{\Phi}_u - (4+v^2)\vec{\Phi}_v \cdot \vec{\Phi}_v\right)$$

$$= \frac{1}{2(4+v^2)}\left(E - (4+v^2)G\right)$$

$$= \frac{1}{2(4+v^2)}\left((4+v^2) - (4+v^2)\right) = 0.$$

Now since $\kappa_1 = \frac{2}{4+v^2}$ and $\kappa_2 = -\frac{2}{4+v^2}$, the Gauss curvature is:

$$\mathrm{K} = \frac{2}{4+v^2}\left(-\frac{2}{4+v^2}\right) = -\frac{4}{(4+v^2)^2} < 0.$$

And the mean curvature is:

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = 0$$

A surface with mean curvature equal to zero everywhere is called a minimal surface. Given a boundary curve, γ, a minimal surface, S, is a surface that has the smallest area among all surfaces with boundry curve γ. The helicoid is an example of a minimal surface.

Ex. A corollary to Euler's theorem says the principal curvatures are the maximum and minimum normal curvature of all curves through a given point on a surface. Show by direct calculation that this is true for the surface in the previous example.

Since any two curves through a point on a surface with parallel velocity vectors have the same normal curvature, we only need to calculate the maximum and minimum normal curvatures of curves of the form:

$$u(t) = u_0 + t$$

$$v(t) = v_0 + \alpha t; \quad \alpha \in \mathbb{R}$$

and the vertical line $u(t) = u_0, \ v(t) = v_0 + t,$

where $\gamma(t) = \vec{\Phi}\big(u(t), v(t)\big)$.

So: $u'(t) = 1$ and $v'(t) = \alpha$ (except for the vertical line).

From the previous example we know that:

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = v^2 + 4 \qquad L = \vec{\Phi}_{uu} \cdot \vec{N} = 0$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = 0 \qquad M = \vec{\Phi}_{uv} \cdot \vec{N} = \frac{2}{\sqrt{4+v^2}}$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = 1 \qquad N = \vec{\Phi}_{vv} \cdot \vec{N} = 0.$$

Plugging in to the formula for normal curvature at $\vec{\Phi}(u_0, v_0)$ we get:

$$\kappa_n = \frac{L\left(\frac{du}{dt}\right)^2 + 2M\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + N\left(\frac{dv}{dt}\right)^2}{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2} = \frac{\frac{4\alpha}{\sqrt{4+v_0^2}}}{v_0^2 + 4 + \alpha^2}$$

$$= \frac{4}{\sqrt{4+v_0^2}}\left(\frac{\alpha}{v_0^2 + 4 + \alpha^2}\right).$$

Now find the max/min of κ_n over all $\alpha \in \mathbb{R}$ and the vertical line $u(t) = u_0$, $v(t) = v_0 + t$.

Through direct calculation we find that:

$$\kappa_n'(\alpha) = \frac{4}{\sqrt{4+v_0^2}}\left[\frac{v_0^2+4-\alpha^2}{\left(v_0^2+4+\alpha^2\right)^2}\right] = 0 \quad \Rightarrow \quad \alpha = \pm\sqrt{v_0^2+4}.$$

By checking the sign of $\kappa_n'(\alpha)$ as we go through the points $\alpha = \pm\sqrt{v_0^2+4}$ we see that $\kappa_n(\alpha)$ has a local minimum at $\alpha = -\sqrt{v_0^2+4}$ and a local maximum at $\alpha = \sqrt{v_0^2+4}$.

Since $\lim_{\alpha\to\pm\infty} \kappa_n(\alpha) = 0$, the local maximum and minimum are global maxima and minima.

$$\kappa_n\left(-\sqrt{v_0^2+4}\right) = \frac{4}{\sqrt{4+v_0^2}}\left(\frac{-\sqrt{v_0^2+4}}{v_0^2+4+(-\sqrt{v_0^2+4})^2}\right)$$

$$= -\frac{2}{v_0^2+4}$$

$$\kappa_n\left(\sqrt{v_0^2+4}\right) = \frac{4}{\sqrt{4+v_0^2}}\left(\frac{\sqrt{v_0^2+4}}{v_0^2+4+(\sqrt{v_0^2+4})^2}\right)$$

$$= \frac{2}{v_0^2+4}.$$

For the vertical line $u(t) = u_0, \;\; v(t) = v_0 + t$, we find $\kappa_n = 0$, so the absolute maximum and minimum of the normal curvature occurs at the principal curvatures.

Ex. Given the Gauss curvature, K, and the mean curvature, H, find the principal curvatures.

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \qquad (*)$$

$$\mathrm{K} = \kappa_1 \kappa_2 \qquad (**)$$

Now solve these equations simultaneously to find κ_1 and κ_2.

From $(**)$, $\kappa_2 = \frac{\mathrm{K}}{\kappa_1}$.

From $(*)$, $2H = (\kappa_1 + \kappa_2) = \kappa_1 + \frac{\mathrm{K}}{\kappa_1}$

$$2H\kappa_1 = \kappa_1^2 + \mathrm{K}$$

$$0 = \kappa_1^2 - 2H\kappa_1 + \mathrm{K}.$$

Now using the quadratic formula:

$$\kappa_1 = \frac{2H \pm \sqrt{4H^2 - 4\mathrm{K}}}{2} = H \pm \sqrt{H^2 - \mathrm{K}}.$$

So we have:

$$\kappa_1 = H + \sqrt{H^2 - \mathrm{K}}$$

$$\kappa_2 = H - \sqrt{H^2 - \mathrm{K}}.$$

In the previous example we had: $H = 0$, $\mathrm{K} = -\frac{4}{(4+\mathrm{v}^2)^2}$,

$$\kappa_1 = 0 + \sqrt{0 + \frac{4}{(4+v^2)^2}} = \frac{2}{4+v^2}$$

$$\kappa_2 = 0 - \sqrt{0 + \frac{4}{(4+v^2)^2}} = \frac{-2}{4+v^2}.$$