Taylor Series

A power series about the point $z_0 \in \mathbb{C}$ is defined as:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \ ; \qquad a_j \in \mathbb{C}.$$

We will focus on power series around $z_0 = 0$, i.e.

$$f(z) = \sum_{j=0}^{\infty} a_j z^j.$$

The general case can be gotten by replacing z by $z - z_0$.

Theorem: If $\sum_{j=0}^{\infty} a_j z^j$ converges for some $z = w$, $|w| = r$, then it converges for all z in $|z| < r$. Moreover, it converges uniformly in $|z| \leq R$, for any $R < r$.

Proof: For $|z| < r$ we have:

$$\left|a_j z^j\right| = \left|a_j r^j\right| \left|\frac{z}{r}\right|^j.$$

But we are assuming that $\sum_{j=0}^{\infty} a_j r^j$ converges thus $\lim_{j \to \infty} a_j r^j = 0$

because the j^{th} term of a convergent series must go to 0 as j goes to ∞.

Thus for j large enough, i.e., there exists a J such that if $j \geq J$ then $\left|a_j r^j\right| < 1$.

So if $j \geq J$:

$$\left|a_j z^j\right| = \left|a_j r^j\right| \left|\frac{z}{r}\right|^j < \left|\frac{z}{r}\right|^j \leq \left(\frac{R}{r}\right)^j.$$

Now take $M = \frac{R}{r} < 1$ (since $R < r$); and $M_j = M^j$.

By the Weierstrass M test $\sum_{j=0}^{\infty} a_j z^j$ converges uniformly in $|z| \leq R$ because:

$$\sum_{j=J}^{\infty} \left|a_j z^j\right| < \sum_{j=J}^{\infty} M^j = \frac{M^J}{1-M} .$$

This theorem says that if a power series converges at a point $z = w$ then it converges uniformly (and hence we can interchange an integral sign with a sum sign) in any closed disk with radius smaller than $|w|$.

Theorem (Taylor Series) Let $f(z)$ be analytic in $|z| \leq R$ then

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \qquad \text{where} \quad a_j = \frac{f^{(j)}(0)}{j!}$$

and the power series converges uniformly in $|z| \leq R_1 < R$.

Outline of Proof: By Cauchy's integral formula we have:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} dw$$

where C is a circle of radius R.

We can then write: $\frac{1}{w-z} = \frac{1}{w\left(1-\frac{z}{w}\right)}$, thus we have:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w} \left(\frac{1}{\left(1-\frac{z}{w}\right)}\right) dw.$$

It's not hard to show that:

$$\frac{1}{\left(1-\frac{z}{w}\right)} = \sum_{j=0}^{\infty} \left(\frac{z}{w}\right)^j \qquad \text{(sum of a geometric series)}$$

and the sum on the right converges uniformly for $\left|\frac{z}{w}\right| < 1$.

So we have:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w}\left(\frac{1}{\left(1-\frac{z}{w}\right)}\right)dw = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w}\left(\sum_{j=0}^{\infty}\left(\frac{z}{w}\right)^j\right)dw$$

$$= \frac{1}{2\pi i}\oint_C f(w)\left(\sum_{j=0}^{\infty}\frac{z^j}{w^{j+1}}\right)dw$$

$$= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\oint_C \frac{f(w)}{w^{j+1}}dw\right)z^j .$$

So if we write: $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then we have:

$$a_j = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w^{j+1}}dw = \frac{f^{(j)}(0)}{j!}$$

by Cauchy's integral formula for derivatives.

Notice that the formula for the coefficients of the Taylor Series is the same as the case of one real variable. If we want the Taylor Series around $z = z_0$ we have:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j; \quad \text{where} \quad a_j = \frac{f^{(j)}(z_0)}{j!} .$$

We have now seen that if a complex function has one derivative in $D \subseteq \mathbb{C}$, it also has an infinite number of derivatives and the Taylor Series of the function converges uniformly to the function inside any disk inside of D. This is in contrast to functions of one real variable where you can have a function with n derivatives, but not $n + 1$ derivatives. In addition, with one real variable you can have a function with an infinite number of derivatives at a point, but its Taylor Series does not converge to the function in any neighborhood of that point (e.g.

$f(x) = e^{-\left(\frac{1}{x^2}\right)}$ if $x \neq 0$

$= 0$ if $x = 0$.)

Ex. Find the Taylor Series around $z_0 = 0$, and the radius of convergence for $f(z) = e^z$.

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \qquad \text{where } a_j = \frac{f^{(j)}(0)}{j!}$$

$$f(z) = e^z \text{ so } f^{(j)}(z) = e^z; \text{ thus } f^{(j)}(0) = 1.$$

$$\Rightarrow a_j = \frac{f^{(j)}(0)}{j!} = \frac{1}{j!}.$$

$f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$ is the Taylor Series for $f(z) = e^z$.

We have already seen that the radius of convergence of this power series is $R = \infty$.

Ex. Find the Taylor Series around $z_0 = 0$, and the radius of convergence for

a. $f(z) = \sin(z)$
b. $f(z) = \frac{1}{1-z}$.

a.

$$\begin{aligned} f(z) &= \sin(z) & f(0) &= 0 \\ f'(z) &= \cos(z) & f'(0) &= 1 \\ f''(z) &= -\sin(z) & f''(0) &= 0 \\ f'''(z) &= -\cos(z) & f'''(0) &= -1 \\ f^{(4)}(z) &= \sin(z) & f^{(4)}(0) &= 0 \end{aligned}$$

Taylor Series: $$f(z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!}$$

Radius of Convergence:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} z^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{(-1)^n z^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{z^2}{(2n+3)(2n+2)} \right| = 0$$

for all $z \in \mathbb{C}$. $\Rightarrow R = \infty$

b. $f(z) = \frac{1}{1-z}$ $\qquad f(0) = 1$

$f'(z) = \frac{1}{(1-z)^2}$ $\qquad f'(0) = 1$

$f''(z) = \frac{2}{(1-z)^3}$ $\qquad f''(0) = 2$

$f'''(z) = \frac{3!}{(1-z)^4}$ $\qquad f'''(0) = 3!$

$f^{(j)}(z) = \frac{j!}{(1-z)^{j+1}}$ $\qquad f^{(j)}(0) = j!$

Taylor Series: $f(z) = \sum_{j=0}^{\infty} \frac{j! z^j}{j!} = \sum_{j=0}^{\infty} z^j$

Radius of Convergence: $\lim_{n\to\infty} \left|\frac{z^{n+1}}{z^n}\right| = |z| < 1.$

So $R = 1$.

Sometimes a Taylor Series can be found by substitution into known Taylor Series.

Ex. Find the Taylor Series for :

a. $f(z) = e^{z^2}$

b. $f(z) = \frac{1}{1+2z^2}$

c. $f(z) = \frac{\sin(z)-z}{z^2}$

a. We know $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$ converges for all $z \in \mathbb{C}$. So

$e^{z^2} = \sum_{j=0}^{\infty} \frac{(z^2)^j}{j!} = \sum_{j=0}^{\infty} \frac{z^{2j}}{j!}$ will also converge for all $z \in \mathbb{C}$.

b. We know $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$ converges for $|z| < 1$. So

$$\frac{1}{1+2z^2} = \frac{1}{1-(-2z^2)} = \sum_{j=0}^{\infty}(-2z^2)^j = \sum_{j=0}^{\infty}(-1)^j 2^j z^{2j}$$

converges for all $|-2z^2| < 1$. This is the same as $|z|^2 < \frac{1}{2}$ or $|z| < \frac{1}{\sqrt{2}}$.

c. $$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots$$

$$\sin(z) - z = -\frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots$$

$$\frac{\sin(z)-z}{z^2} = -\frac{z}{3!} + \frac{z^3}{5!} + \cdots + \frac{(-1)^n z^{2n-1}}{(2n+1)!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^j z^{2n-1}}{(2n+1)!}$$

This power series converges for $|z| < \infty$ by the ratio test, however, we can only say it equals $\frac{\sin(z)-z}{z^2}$ for $z \neq 0$.

Taylor Series behave in many ways like polynomials. We have already seen that we can integrate them term by term, i.e.

$$\int f(z)dz = \int \sum_{j=0}^{\infty}(a_j z^j)dz = \sum_{j=0}^{\infty} \int (a_j z^j)dz = \sum_{j=0}^{\infty} \frac{a_j}{(j+1)} z^{j+1} + C$$

where C is a constant.

We can also add series term by term.

Ex. Find the Taylor Series for $f(z) = \cosh(z)$.

$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$ and $e^{-z} = \sum_{j=0}^{\infty} \frac{(-z)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{j!}$ where both power series converge for all $z \in \mathbb{C}$.

$$\cosh(z) = \frac{e^z+e^{-z}}{2} = \frac{1}{2}[\sum_{j=0}^{\infty}\frac{z^j}{j!} + \sum_{j=0}^{\infty}\frac{(-1)^j z^j}{j!}]$$

$$= \frac{1}{2}[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots + \frac{z^n}{n!} + \cdots\right)$$
$$+ \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots + \frac{(-1)^n z^n}{n!} + \cdots\right)]$$

$$= \frac{1}{2}(2 + \frac{2z^2}{2!} + \frac{2z^4}{4!} + \cdots + \frac{2z^{2n}}{(2n)!} + \cdots)$$

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}.$$

This power series converges for all $z \in \mathbb{C}$ because each of the power series we added to get it do (If you add two power series that converge on different sets, the sum will converge on the intersection of those sets.).

We can also differentiate Taylor Series term by term to get the derivative of a function.

Theorem: Let $f(z)$ be analytic for $|z| \leq R$. Then the series obtained by differentiating the Taylor series termwise converges uniformly to $f'(z)$ in $|z| \leq R_1 < R$.

Proof. We know that if $f(z)$ is analytic in a disk D, $|z| \leq R$, then so is $f'(z)$. Since $f'(z)$ is analytic, it has a convergent Taylor series:

$$f'(z) = \sum_{j=0}^{\infty} c_j z^j; \quad \text{where} \quad c_j = \frac{(f')^{(j)}(0)}{j!} = \frac{f^{(j+1)}(0)}{j!}.$$

The Taylor series for $f(z)$ is:

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j.$$

If we differentiate the power series for $f(z)$ term by term we get:

$$\sum_{j=1}^{\infty} \frac{f^{(j)}(0)}{(j-1)!} z^{j-1} = \sum_{j=0}^{\infty} \frac{f^{(j+1)}(0)}{j!} z^j.$$

Which is the Taylor series for $f'(z)$.

Notice that the Taylor series of a function $f(z)$ must be unique. Suppose there were 2 Taylor series for $f(z)$, $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=0}^{\infty} b_j z^j$.

Let $g(z) = \sum_{j=0}^{\infty} a_j z^j - \sum_{j=0}^{\infty} b_j z^j = f(z) - f(z) = 0.$

The Taylor series for $g(z) = 0$ is the series $\sum_{j=0}^{\infty} c_j z^j$ where $c_j = 0$ for all j.

But $\sum_{j=0}^{\infty} c_j z^j = \sum_{j=0}^{\infty} a_j z^j - \sum_{j=0}^{\infty} b_j z^j = \sum_{j=0}^{\infty} (a_j - b_j) z^j$;

so: $0 = c_j = a_j - b_j$ for all j, and thus $a_j = b_j$ for all j.

Thus $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=0}^{\infty} b_j z^j$ are the same power series.

Ex. Find the Taylor series for $g(z) = \cos(z)$ from the Taylor series for

$f(z) = \sin(z)$.

Since $g(z) = f'(z)$, we can just differentiate the Taylor series for $f(z) = \sin(z)$ term by term to get the Taylor series for $g(z) = \cos(z)$.

$$f(z) = \sin(z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!}; \quad \text{thus}$$

$$g(z) = \cos(z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j)!}\,.$$

Since the radius of convergence of the power series for $\sin(z)$ is $R = \infty$ then so is the radius of convergence of the derivative of the power series of $\sin(z)$.

Thus the radius of convergence of the power series for $\cos(z)$ is also $R = \infty$.

Ex. Find the Taylor series for

a. $\frac{z^2}{2-z^3}$; $|z| < \sqrt[3]{2}$

b. $\frac{\sin(z^2)-z^2}{z^4}$; $0 < |z| < \infty$.

a. $$\frac{z^2}{2-z^3} = \frac{z^2}{2}\left[\frac{1}{\left(1-\frac{z^3}{2}\right)}\right].$$

We know that $\frac{1}{1-w} = \sum_{j=0}^{\infty} w^j$ converges for $|w| < 1$. So if $w = \frac{z^3}{2}$:

$$\frac{z^2}{2}\left[\frac{1}{\left(1-\frac{z^3}{2}\right)}\right] = \frac{z^2}{2}\sum_{j=0}^{\infty}\left(\frac{z^3}{2}\right)^j = \frac{z^2}{2}\sum_{j=0}^{\infty}\frac{z^{3j}}{2^j}$$

$$= \sum_{j=0}^{\infty}\frac{z^{3j+2}}{2^{j+1}}$$

which converges for $\left|\frac{z^3}{2}\right| < 1$ or $|z| < \sqrt[3]{2}$.

Thus $\frac{z^2}{2-z^3} = \sum_{j=0}^{\infty}\frac{z^{3j+2}}{2^{j+1}}$; and converges for $|z| < \sqrt[3]{2}$.

b. We know that $\sin(z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!}$; $|z| < \infty$ so

$$\sin(z^2) = \sum_{j=0}^{\infty} \frac{(-1)^j (z^2)^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} \frac{(-1)^j z^{4j+2}}{(2j+1)!}; \quad |z| < \infty$$

$$\sin(z^2) = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + \cdots + \frac{(-1)^j z^{4j+2}}{(2j+1)!} + \cdots$$ So

$$\sin(z^2) - z^2 = -\frac{z^6}{3!} + \frac{z^{10}}{5!} + \cdots + \frac{(-1)^j z^{4j+2}}{(2j+1)!} + \cdots \; ; \quad |z| < \infty.$$

$$\frac{\sin(z^2) - z^2}{z^4} = -\frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots + \frac{(-1)^j z^{4j-2}}{(2j+1)!} + \cdots$$

$$= z^2[-\frac{1}{3!} + \frac{z^4}{5!} - \frac{z^8}{7!} + \cdots + \frac{(-1)^j z^{4j-4}}{(2j+1)!} + \cdots]; \; 0 < |z| < \infty \; .$$

So $\frac{\sin(z^2) - z^2}{z^4}$ approaches 0 like z^2 as z goes to 0.

Ex. Evaluate $\oint_C \left(\frac{\sin(z^2)-z^2}{z^7}\right)dz$; where C is the unit circle $|z| = 1$.

From the previous example we know that:

$$\frac{\sin(z^2)-z^2}{z^4} = \sum_{j=1}^{\infty} \frac{(-1)^j z^{4j-2}}{(2j+1)!};$$ So

$$\frac{\sin(z^2)-z^2}{z^7} = \frac{1}{z^3}\sum_{j=1}^{\infty} \frac{(-1)^j z^{4j-2}}{(2j+1)!} = \sum_{j=1}^{\infty} \frac{(-1)^j z^{4j-5}}{(2j+1)!}$$

$$= -\frac{1}{3!}\frac{1}{z} + \frac{z^3}{5!} - \frac{z^7}{7!} + \cdots + \frac{(-1)^j z^{4j-5}}{(2j+1)!} + \cdots$$

We saw earlier that for C, the unit circle:

$$\oint_C z^j dz = 0 \qquad if\ j \neq -1$$
$$= 2\pi i \qquad if\ j = -1;$$ So

$$\oint_C \frac{\sin(z^2)-z^2}{z^7} dz = \oint_C \sum_{j=1}^{\infty} \frac{(-1)^j z^{4j-5}}{(2j+1)!} dz = \sum_{j=1}^{\infty} \oint_C \frac{(-1)^j z^{4j-5}}{(2j+1)!} dz$$

$$= \oint_C \left(-\frac{1}{3!}\frac{1}{z} + \frac{z^3}{5!} - \frac{z^7}{7!} + \cdots + \frac{(-1)^j z^{4j-5}}{(2j+1)!} + \cdots\right)dz$$

$$= \oint_c -\frac{1}{3!}\frac{1}{z} dz + \oint_C \left(\frac{z^3}{5!} - \frac{z^7}{7!} + \cdots + \frac{(-1)^j z^{4j-5}}{(2j+1)!} + \cdots\right)dz$$

$$= -\frac{1}{6}(2\pi i) + 0 = -\frac{\pi i}{3}.$$

From the Taylor series for $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$ we know

$$\frac{1}{1+z} = \sum_{j=0}^{\infty}(-1)^j z^j; \quad |z| < 1.$$

This series only converges, and converges to $\frac{1}{1+z}$, when $|z| < 1$.

For example, even though we can substitute $z = 2$ into $\frac{1}{1+z}$ and get $\frac{1}{1+2} = \frac{1}{3}$, substituting $z = 2$ into the power series gives us :

$$\sum_{j=0}^{\infty}(-1)^j (2)^j = 1 - 2 + 4 - 8 + 16 + \cdots$$

which doesn't converge. However, we can find a series expression for $\frac{1}{1+z}$ that will converge for $z = 2$ (and converge to $\frac{1}{3}$).

This is how we do it. Notice that:

$$\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right).$$

Let's look at the term $\frac{1}{1+\frac{1}{z}}$. We know if $\left|\frac{1}{z}\right| < 1$, i.e. $|z| > 1$ then

$$\frac{1}{1+\frac{1}{z}} = \sum_{j=0}^{\infty}(-1)^j (\frac{1}{z})^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{z^j}.$$

Now we can create a series for $\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right)$:

$$\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) = \frac{1}{z}\sum_{j=0}^{\infty} \frac{(-1)^j}{z^j} = \sum_{j=0}^{\infty} \frac{(-1)^j}{z^{j+1}}; \quad |z| > 1$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots + \frac{(-1)^j}{z^{j+1}} + \cdots .$$

Now notice when we plug $z = 2$ into this series we get:

$$\frac{1}{1+2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots \ .$$

This is a geometric series with a ratio between term of $-\frac{1}{2}$, thus the sum is:

$$S = \frac{\frac{1}{2}}{1-(-\frac{1}{2})} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}\,.$$

However, notice the series $\sum_{j=0}^{\infty} \frac{(-1)^j}{z^{j+1}} = \sum_{j=0}^{\infty}(-1)^j z^{-(j+1)}$ is NOT a power series in z because the exponents of z are not all non-negative (in fact, they are all negative). This is called a **Laurent series.**

Theorem (comparison test): Let the series $\sum_{j=0}^{\infty} a_j z^j$ converge for $|z| < R$. If $|b_j| \le |a_j|$ for $j \ge J$ for some J, then the series $\sum_{j=0}^{\infty} b_j z^j$ converges for $|z| < R$.

Proof: Suppose $\sum_{j=0}^{\infty} a_j z^j$ converges for $z = c$, where $|c| < R$.
Since $\sum_{j=0}^{\infty} a_j c^j$ converges, $\lim_{n\to\infty} a_j c^j = 0$.

Thus there exists a J such that if $j \ge J$ then $|a_j c^j| < 1$.
For any z such that $|z| < |c|$ and $j \ge J$:

$$|b_j z^j| \le |a_j z^j| = |a_j c^j| \left|\frac{z}{c}\right|^j < \left|\frac{z}{c}\right|^j.$$

Now we use the Weierstrass M test to show that $\sum_{j=0}^{\infty} b_j z^j$ converges.
Let $M_j = \left|\frac{z}{c}\right|^j$ then $|b_j z^j| < \left|\frac{z}{c}\right|^j = M_j$, and

$$\sum_{j=J}^{\infty} M_j = \sum_{j=J}^{\infty} \left|\frac{z}{c}\right|^j.$$

Since $\left|\frac{z}{c}\right| < 1$, $\sum_{j=J}^{\infty}\left|\frac{z}{c}\right|^{j}$ converges (It's a geometric series with the absolute value of the ratio less than one.).

Now since $\sum_{j=J}^{\infty} M_j$ converges so does $\sum_{j=0}^{\infty} b_j z^j$.

Ex. Show $\sum_{n=0}^{\infty}\frac{z^n}{(n!)^3}$ converges for all $z \in \mathbb{C}$.

$$\frac{1}{(n!)^3} \leq \frac{1}{n!} \text{ thus } \left|\frac{z^n}{(n!)^3}\right| \leq \left|\frac{z^n}{n!}\right|.$$

We know that $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ converges to e^z for all $z \in \mathbb{C}$.

So by the comparison test $\sum_{n=0}^{\infty}\frac{z^n}{(n!)^3}$ converges for all $z \in \mathbb{C}$.

Theorem: Let $f(z)$ and $g(z)$ be analytic functions in a common domain D. If $f(z) = g(z)$ in some subdomain $D' \subseteq D$ or on a curve C interior to D, then $f(z) = g(z)$ everywhere in D.

Corollary: If $f(z)$ is analytic in a domain D and $f(z) = 0$ on some subdomain $D' \subseteq D$ or on a curve C interior to D, then $f(z) = 0$ on D.

Theorem: Let $f(z)$ be analytic and not identically zero in a domain D with $f(z_0) = 0,\ z_0 \in D$. Then z_0 is an isolated zero; that is, there is a neighborhood about $z = z_0$ where $f(z)$ is non-zero except at $z = z_0$.

Proof: Since $f(z)$ is analytic in D it has a Taylor series about $z = z_0$:

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j.$$

If $f(z)$ has a zero of order m at $z = z_0$ then:

$$f(z) = \sum_{j=m}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j.$$

m must be finite otherwise $f(z) = 0$ for all $z \in D$ (and it's not by assumption).

Thus we can write:

$$f(z) = (z - z_0)^m g(z)$$

where $g(z)$ has a Taylor series around $z = z_0$ and $g(z_0) \neq 0$.

Since $g(z)$ is analytic (it has a Taylor series that converges to it) in D, it must also be continuous in D.

Hence we can find a neighborhood around $z = z_0$ such that $g(z) \neq 0$.

But in that neighborhood $f(z) \neq 0$ since $f(z) = (z - z_0)^m g(z)$.

Hence $z = z_0$ is an isolated zero of $f(z)$.