Surface Integrals of Scalar Functions

Let $\vec{\Phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a surface S and D an Elementary Region (ie, $y \in [c,d]$ and $h_1(y) \le x \le h_2(y)$, and/or $x \in [a,b]$ and $k_1(x) \le y \le k_2(x)$): $\vec{\Phi}(u,v) =< x(u,v),\ y(u,v),\ z(u,v) >$.

Def. If $f(x,y,z)$ is a real-valued continuous function on a parametrized surface S, we define the integral of $f(x,y,z)$ over S to be:

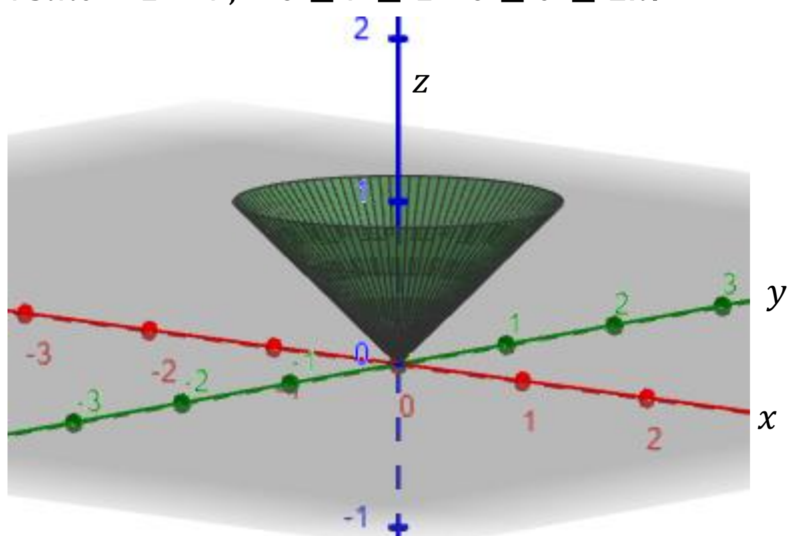
$$\iint_S \boldsymbol{f(x,y,z)dS} = \iint_D \boldsymbol{f(\vec{\Phi}(u,v))\left|\vec{T}_u \times \vec{T}_v\right|dudv}.$$

Notice that if $f(x,y,z) = 1$, then this integral equals the surface area of S.

The motivation for this definition is similar to that for surface area.

Ex. Evaluate $\iint_S f dS$ when $f(x,y,z) = x^2 + y^2 + z$ and S is the portion of the cone defined by: $x = r\cos\theta \quad y = r\sin\theta \quad z = r; \quad 0 \le r \le 1 \quad 0 \le \theta \le 2\pi$.

We saw from an earlier example:



$$\vec{\Phi}(r,\theta) =< r\cos\theta, r\sin\theta, r >$$

$$\vec{T}_r =< \cos\theta, \sin\theta, 0 >$$

$$\vec{T}_\theta =< -r\sin\theta, r\cos\theta, 1 >$$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta\vec{i} - r\sin\theta\vec{j} + r\vec{k}\ ;$$

So we have:

$$\left|\vec{T}_r \times \vec{T}_\theta\right| = \sqrt{r^2cos^2\theta + r^2sin^2\theta + r^2}$$

$= |r|\sqrt{2}$. $(r \geq 0$, so don't need | |).

$$f\left(\vec{\Phi}(r,\theta)\right) = x^2 + y^2 + z = r^2cos^2\theta + r^2sin^2\theta + r.$$

$$\iint_S \ f dS = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r^2cos^2\theta + r^2sin^2\theta + r)r\sqrt{2}drd\theta$$

$$= \sqrt{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r^2 + r)rdrd\theta = \sqrt{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r^3 + r^2)drd\theta$$

$$= \sqrt{2}\int_{\theta=0}^{2\pi}d\theta\int_{r=0}^{1}(r^3 + r^2)dr = \sqrt{2}(2\pi)(\frac{r^4}{4} + \frac{r^3}{3}\Big|_0^1 = \frac{7\sqrt{2}\pi}{6}.$$

Ex. Compute the surface integral $\iint_S \ x^2 dS$, where S is the unit sphere.

$x = cos\theta sin\phi \quad y = sin\theta sin\phi \quad z = cos\phi; \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$

$$\vec{\Phi}(\phi,\theta) =< cos\theta sin\phi, sin\theta sin\phi, cos\phi >; \quad 0 \leq \phi \leq \pi \quad 0 \leq \theta \leq 2\pi.$$

$$\vec{T}_\phi =< cos\theta cos\phi, sin\theta cos\phi, -sin\phi >$$

$$\vec{T}_\theta =< -sin\theta sin\phi, cos\theta sin\phi, 0 >$$

$$\vec{T}_\phi \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ cos\theta cos\phi & sin\theta cos\phi & -sin\phi \\ -sin\theta sin\phi & cos\theta sin\phi & 0 \end{vmatrix}$$

$$\vec{T}_\phi \times \vec{T}_\theta = (cos\theta \sin^2\phi)\vec{\imath} + (sin\theta \sin^2\phi)\vec{\jmath} + (sin\,\phi cos\phi)\vec{k}$$

$$\left|\vec{T}_\phi \times \vec{T}_\theta\right| = \sqrt{[\cos^2\theta \sin^4\phi + \sin^2\theta \sin^4\phi + \sin^2\phi \cos^2\phi]}$$

$$= \sqrt{\sin^4\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)} = sin\phi.$$

$$f\left(\vec{\Phi}(r,\theta)\right) = x^2 = \cos^2\theta \sin^2\phi.$$

$$\iint_S \; x^2 dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} (\cos^2\theta \sin^2\phi)\, sin\phi \;\, d\phi d\theta$$

$$= (\int_{\theta=0}^{2\pi} (\cos^2\theta) d\theta)(\int_{\phi=0}^{\phi=\pi} (\sin^3\phi) d\phi)$$

$$= [\int_{\theta=0}^{2\pi} \left(\frac{1}{2} + \frac{1}{2} cos2\theta\right) d\theta]\,[\int_{\phi=0}^{\phi=\pi} (1 - cos^2\phi)(sin\,\phi) d\phi]$$

$$= \left[\left(\frac{1}{2}\theta + \frac{1}{4} sin2\theta\right)\Big|_{0}^{2\pi}\right][(-cos\theta + \frac{1}{3}\cos^3\theta)|_{0}^{\pi}$$

$$= \pi\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = \frac{4\pi}{3}.$$

<u>Surface Integrals when S is given by $z = g(x, y)$</u>

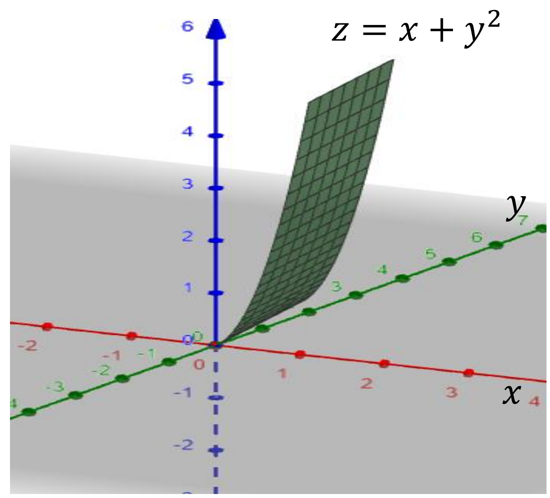
As we saw with surface area, when $z = g(x, y)$ we can parametrize the surface:

$x = u \quad y = v \quad z = g(u, v)$ and

$\left|\vec{T}_u \times \vec{T}_v\right| = \sqrt{1 + (g_u)^2 + (g_v)^2} = \sqrt{1 + (g_x)^2 + (g_y)^2}$; so we have:

$$\boldsymbol{\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + (g_x)^2 + (g_y)^2}\, dxdy}$$

Ex. Evaluate $\iint_S y dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1 \quad 0 \le y \le 2$.



In this case, $g_x = 1$ and $g_y = 2y$.

$\iint_S y dS = \int_{y=0}^{2} \int_{x=0}^{1} y\sqrt{1 + 1^2 + (2y)^2}\, dxdy$

$\iint_S y dS = \int_{y=0}^{2} \int_{x=0}^{1} y(2 + 4y^2)^{\frac{1}{2}}\, dxdy$

$= \int_{x=0}^{1} dx \int_{y=0}^{2} y(2 + 4y^2)^{\frac{1}{2}} dy$

$= (1) \int_{y=0}^{2} y(2 + 4y^2)^{\frac{1}{2}} dy$ (let $u = 2 + 4y^2$, $\frac{1}{8} du = ydy$)

$= \frac{13\sqrt{2}}{3}$.

To find the center of mass of a surface S, where $\rho(x, y, z)$ =density at (x, y, z).

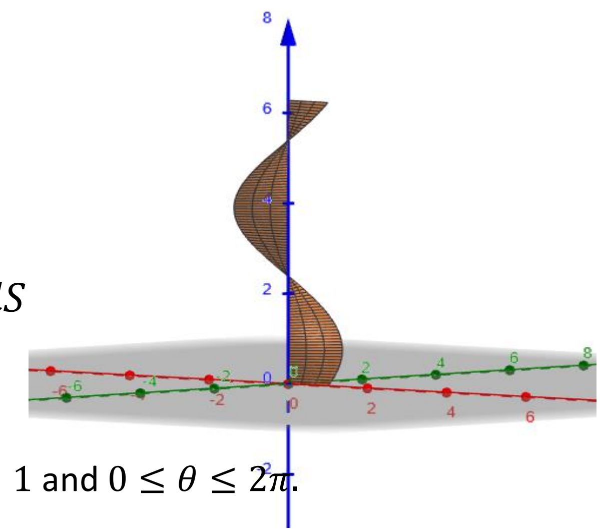
m = mass= $\iint_S \rho(x, y, z)dS$ and $(\bar{x}, \bar{y}, \bar{z})$ is the center of mass.

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x, y, z)dS$$

$$\bar{y} = \frac{1}{m}\iint_S y\rho(x, y, z)dS$$

$$\bar{z} = \frac{1}{m}\iint_S z\rho(x, y, z)dS$$

Ex. Suppose a Helicoid, S, is parametrized by $\vec{\Phi}(r, \theta) =< r\cos\theta, r\sin\theta, \theta >$; where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Suppose the density at $(x, y, z) \in S$ equals the distance to the z-axis. Find the mass and write down definite integrals representing the center of mass $(\bar{x}, \bar{y}, \bar{z})$.



In this case $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

m = mass= $\iint_S \rho(x, y, z)dS = \iint_S \sqrt{x^2 + y^2}dS$

$\vec{\Phi}(r, \theta) =< r\cos\theta, r\sin\theta, \theta >$; where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

as we have seen in a previous example:

$\vec{T}_r =< \cos\theta, \sin\theta, 0 >$ $\qquad$ $\vec{T}_\theta =< -r\sin\theta, r\cos\theta, 1 >$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = \sin\theta\vec{\imath} - \cos\theta\vec{\jmath} + r\vec{k}\ .$$

$$|\vec{T}_r \times \vec{T}_\theta| = \sqrt{1 + r^2}\ .$$

We also know that $\sqrt{x^2+y^2} = \left(\sqrt{r^2cos^2\theta + r^2sin^2\theta}\right) = r;$ since $r \geq 0.$

$m =$ mass$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\sqrt{1+r^2}drd\theta.$

$= \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{1} r(1+r^2)^{\frac{1}{2}}\, dr$

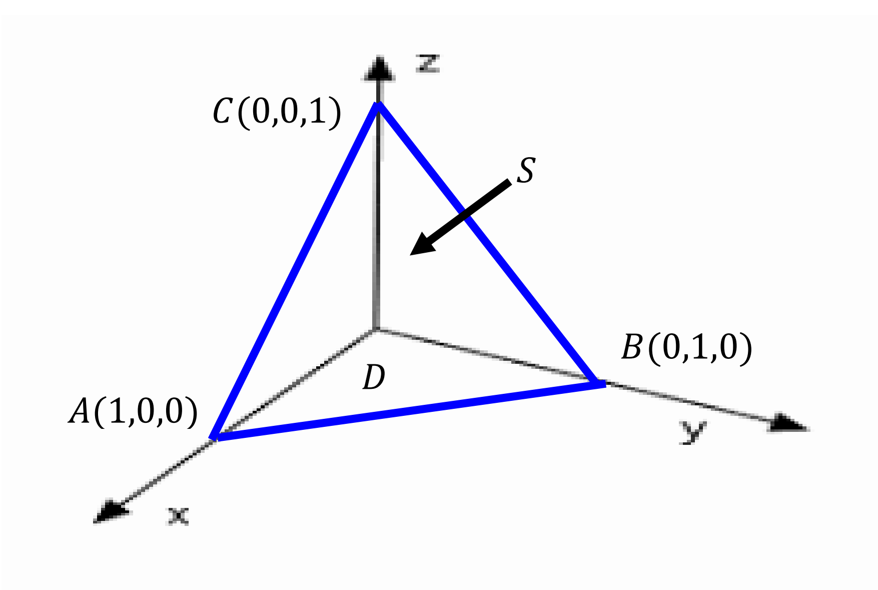
$= (2\pi)(\frac{1}{3}(1+r^2)^{\frac{3}{2}} \Big|_0^1 = \frac{2\pi}{3}(2\sqrt{2}-1).$

$\bar{x} = \frac{1}{m}\iint_S \; x\rho(x,y,z)dS = \frac{1}{m}\int_{\theta=0}^{2\pi} \int_{r=0}^{1} (rcos\theta)(r\sqrt{1+r^2})drd\theta$

$\bar{y} = \frac{1}{m}\iint_S \; y\rho(x,y,z)dS = \frac{1}{m}\int_{\theta=0}^{2\pi} \int_{r=0}^{1} (rsin\theta)(r\sqrt{1+r^2})drd\theta$

$\bar{z} = \frac{1}{m}\iint_S \; z\rho(x,y,z)dS = \frac{1}{m}\int_{\theta=0}^{2\pi} \int_{r=0}^{1} (\theta)(\,r\sqrt{1+r^2})drd\theta\,.$

Ex. Find $\iint_S \; ydS$, where S is bounded by the triangle with vertices $A(1,0,0)$, $B(0,1,0), C(0,0,1)$.



First find the equation of the plane that contains these 3 points.

$\overrightarrow{AB} =< -1, 1, 0 >$ $\qquad\qquad$ $\overrightarrow{AC} =< -1, 0, 1 >$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \vec{\imath} + \vec{\jmath} + \vec{k}.$$

$\vec{N} = \vec{\imath} + \vec{\jmath} + \vec{k} =< 1, 1, 1 >$ is perpendicular to the plane containing $A, B,$ and C.

Using $\vec{N}$ and the point $(1,0,0)$, an equation of this plane is:

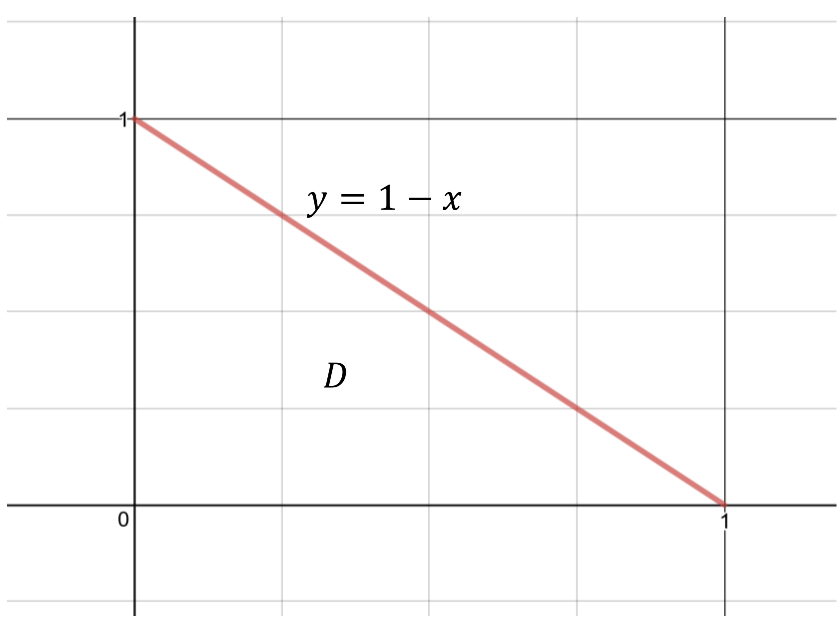
$$(x - 1) + y + z = 0$$

Solving this equation for z: $\quad z = -x - y + 1$, so $\quad z_x = -1 \qquad z_y = -1$.

$$\iint_S y dS = \iint_D y\sqrt{1 + (z_x)^2 + (z_y)^2} dxdy$$

$$= \iint_D y\sqrt{1 + (-1)^2 + (-1)^2}\, dxdy = \iint_D y\sqrt{3}\, dxdy.$$

D is bounded by the x –axis, the y –axis, and the line $y = -x + 1$.

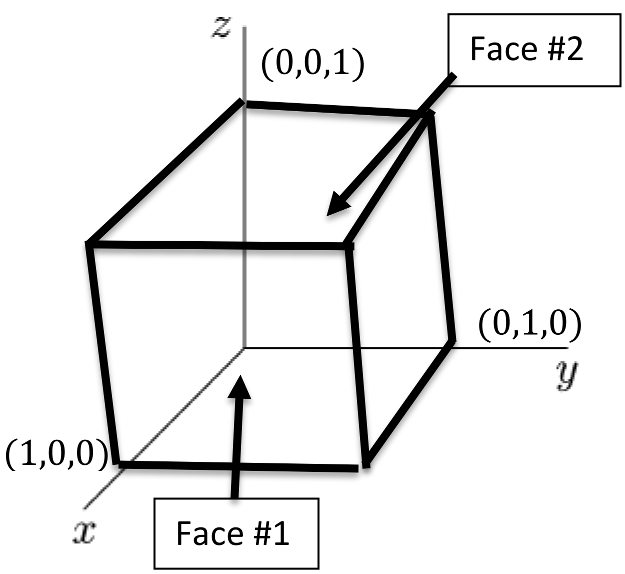


$$\iint_S y\,dS = \int_{x=0}^{x=1}\int_{y=0}^{y=-x+1}(y\sqrt{3})\,dy\,dx$$

$$= \int_{x=0}^{x=1}\sqrt{3}\left(\frac{1}{2}\right)y^2 \Big|_{y=0}^{y=-x+1} dx$$

$$= \int_{x=0}^{x=1}\frac{\sqrt{3}}{2}(-x+1)^2dx = \frac{\sqrt{3}}{2}\left(-\frac{1}{3}\right)(-x+1)^3 \Big|_0^1 = \frac{\sqrt{3}}{6}.$$

Ex. Evaluate $\iint_S x^2 dS$, where S is the boundary of the cube [0,1]x[0,1]x[0,1].



The surface S is made up of the 6 faces of the cube. We must parametrize each face of the cube.

Face #1: $x = 1,\ 0 \le y \le 1,\ 0 \le z \le 1;$ ie $\vec{\Phi}(u, v) =< 1, u, v >;$
$0 \le u, v \le 1$
$\vec{T}_u =< 0,1,0 >, \quad \vec{T}_v =< 0,0,1 >; \quad \vec{T}_u \times \vec{T}_v = \vec{\imath};$ so $\left|\vec{T}_u \times \vec{T}_v\right| = 1.$

$\iint_{Face\ 1} x^2 dS = \int_{z=0}^{z=1} \int_{y=0}^{y=1} 1^2\, dydz = 1.$

Face #2: $x = 0,\ 0 \le y \le 1,\ 0 \le z \le 1;$ ie $\vec{\Phi}(u, v) =< 0, u, v >;$
$0 \le u, v \le 1.$

$\iint_{Face\ 2} x^2 dS = \int_{z=0}^{z=1} \int_{y=0}^{y=1} 0^2\, dydz = 0.$

$Face$ #3: $y = 1,\ 0 \le x \le 1,\ 0 \le z \le 1;$ ie $\vec{\Phi}(u, v) =< u, 1, v >;$
$0 \le u, v \le 1.$

$\left|\vec{T}_u \times \vec{T}_v\right| = 1$

so $\iint_{Face\ 3} x^2 dS = \int_{z=0}^{z=1} \int_{x=0}^{x=1} x^2\, dxdz = \int_{z=0}^{z=1} dz \int_{x=0}^{x=1} x^2 dx = \frac{1}{3}.$

Faces 4-6 work out just the same as face 3 so the integral for each face equals $\frac{1}{3}$.

$\iint_S \ x^2 dS = 1 + 0 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{7}{3}.$