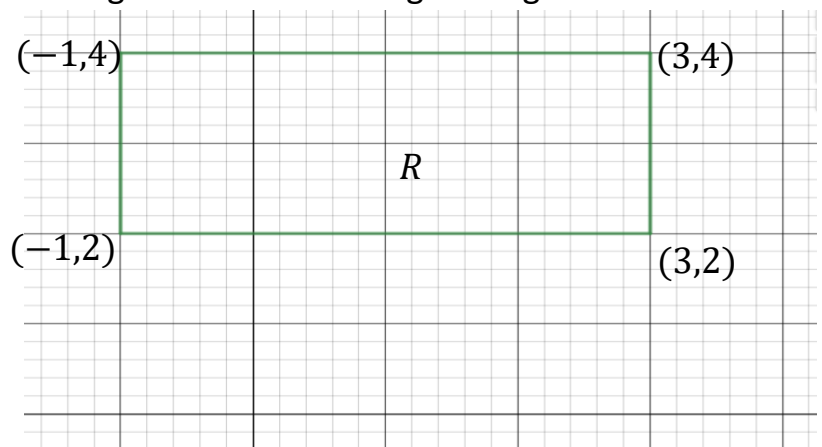A Quick Review of Multiple Integration

<u>Double Integration</u>

If the x and y endpoints of integration are all constants (i.e., just numbers, no variables), then you are integrating over a rectangular region in the plane whose sides are parallel to the x and y axes.

Ex. $\int_{y=2}^{y=4} \int_{x=-1}^{x=3} 2xy^2 dxdy,$ is an integral over the rectangular region R

$-1 \leq x \leq 3$ and $2 \leq y \leq 4$

To evaluate this integrate, we first integrate with respect to x, holding y constant, substitute the values of x, and then integrate with respect to y.

$$\int_{y=2}^{y=4} \int_{x=-1}^{x=3} 2xy^2 dxdy = \int_{y=2}^{y=4} x^2 y^2 \mid_{x=-1}^{x=3} dy = \int_{y=2}^{y=4} y^2(3^2 - (-1)^2)dy$$

$$= \int_{y=2}^{y=4} 8y^2 \, dy = \frac{8}{3}y^3 \mid_{y=2}^{y=4} = \frac{8}{3}(4^3 - 2^3) = \frac{448}{3}.$$

When the endpoints of integration are all constants and the function you are integrating can be written as $f(x,y) = g(x)h(y)$, then

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)dxdy = \int_{y=c}^{y=d} \int_{x=a}^{x=b} g(x)h(y)dxdy.$$

$$= (\int_{y=c}^{y=d} h(y)dy)(\int_{x=a}^{x=b} g(x)dx).$$

Ex. $\int_{y=2}^{y=4} \int_{x=-1}^{x=3} 2xy^2 dxdy = (\int_{y=2}^{y=4} y^2 dy)(\int_{x=-1}^{x=3} 2xdx)$

$$= \left(\frac{1}{3}y^3 \Big|_{y=2}^{y=4}\right)(x^2 \Big|_{x=-1}^{x=3}) = \frac{1}{3}(64-8)(9-1) = \frac{448}{3}.$$

When the region we are integrating over is not a rectangle, the endpoints in x and y will not all be constant. We have 2 methods we can consider in this case.

1. If the region D is bounded below by $y = g(x)$ and above by $y = h(x)$, and along the sides by $x = a$ and $x = b$, we have:
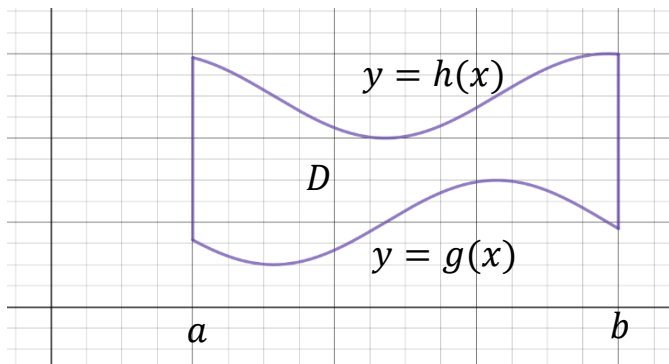
$$\iint_D f(x,y)dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)dydx$$

2. If the region D is bounded on the left side by $x = g(y)$ and on the right side by $x = h(y)$, and below by $y = c$ and above by $y = d$, then we have:

d

$x = g(y)$ $x = h(y)$

D

c
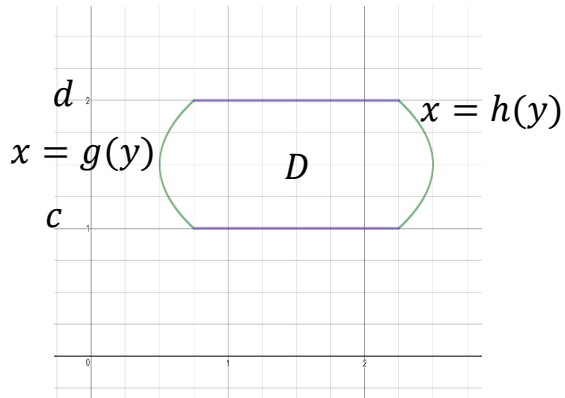
$$\iint_D f(x,y)dA = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x,y)dxdy.$$

Note: **If $f(x,y) = 1$, then $\iint_D f(x,y)dA = \iint_D 1\,dA =$Area of the region D.**

Ex. Evaluate $\iint_D x^2 y \, dA$, where D is the region bounded by the curves $y = x^2$ and $y = 2x$.

In this case we can solve this using either method 1 or method 2.

Start by graphing the curves that bound D. Find the points of intersection of the curves (in this case setting $x^2 = 2x$ we find $x = 0$ and $x = 2$). The points of intersection are (0,0) and (2,4).

$$y = 2x \text{ or } x = \frac{y}{2}$$

$$D$$

$$y = x^2 \text{ or } x = \sqrt{y}$$
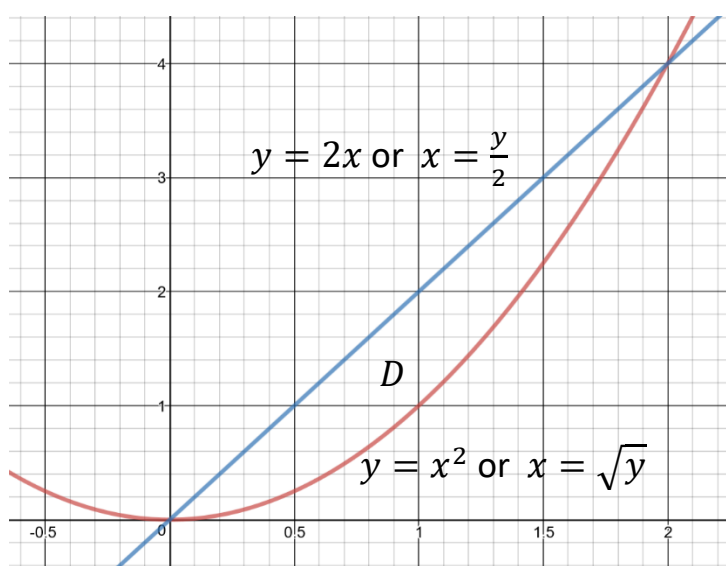
Method 1:

The bottom curve is $y = x^2$ and the top curve is $y = 2x$.

$$\iint_D x^2 y \, dA = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} x^2 y \, dy \, dx$$

$$= \int_{x=0}^{x=2} \frac{x^2}{2} y^2 \Big|_{y=x^2}^{y=2x} dx$$

$$= \int_{x=0}^{x=2} \frac{x^2}{2} \left((2x)^2 - (x^2)^2\right) dx$$

$$= \int_{x=0}^{x=2} \left(2x^4 - \frac{1}{2}x^6\right) dx = \left(\frac{2}{5}x^5 - \frac{1}{14}x^7\right)\Big|_{x=0}^{x=2} = \frac{128}{35}.$$
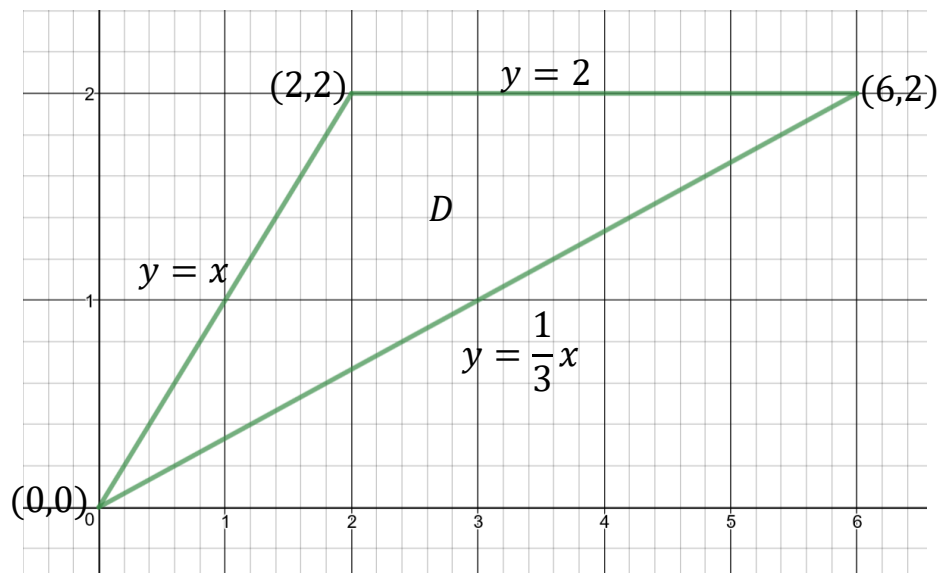
Method 2:

The left curve is $x = \frac{y}{2}$ and the right curve is $x = \sqrt{y}$.

$$\iint_D x^2 y \, dA = \int_{y=0}^{y=4} \int_{x=\frac{y}{2}}^{x=\sqrt{y}} x^2 y \, dx \, dy = \int_{y=0}^{y=4} \frac{x^3}{3} y \Big|_{x=\frac{y}{2}}^{x=\sqrt{y}} dy$$

$$= \int_{y=0}^{y=4} [\frac{y^{\frac{3}{2}}}{3}(y) - \left(\frac{y^3}{24}\right) y] \, dy$$

$$= \int_{y=0}^{y=4} (\frac{1}{3} y^{\frac{5}{2}} - \frac{1}{24} y^4) \, dy$$

$$= \left(\left(\frac{1}{3}\right)\left(\frac{2}{7}\right) y^{\frac{7}{2}} - \frac{1}{120} y^5\right) \Big|_{y=0}^{y=4} = \frac{128}{35}.$$

Ex. Evaluate $\iint_D 2xy \, dA$ where D is the region inside the triangle with vertices $(0,0), (2,2), (6,2)$.

First find the equations of the 3 sides:

$y = \frac{1}{3} x, \quad y = x, \quad$ and $y = 2.$

Method 1.

Notice that the top curve switches at $x = 2$ (from $y = x$ to $y = 2$). Thus we have to break the integral up into 2 pieces if we want to use this method.

$$\iint_D 2xy\,dA = \int_{x=0}^{x=2} \int_{y=\frac{1}{3}x}^{y=x} 2xy\,dy\,dx + \int_{x=2}^{x=6} \int_{y=\frac{1}{3}x}^{y=2} 2xy\,dy\,dx$$

$$= \int_{x=0}^{x=2} xy^2|_{y=\frac{1}{3}x}^{y=x}\,dx + \int_{x=2}^{x=6} xy^2|_{y=\frac{1}{3}x}^{y=2}\,dy$$

$$= \int_{x=0}^{x=2}\left(x^3 - \frac{x^3}{9}\right)dx + \int_{x=2}^{x=6}\left(4x - \frac{x^3}{9}\right)dx$$

$$= \int_{x=0}^{x=2}\frac{8}{9}x^3\,dx + \int_{x=2}^{x=6}\left(4x - \frac{x^3}{9}\right)dx$$

$$= \frac{2}{9}x^4|_{x=0}^{x=2} + \left(2x^2 - \frac{1}{36}x^4\right)|_{x=2}^{x=6} = 32\ .$$

Method 2 (the easier way for this problem).

The left curve is $x = y$ and the right curve is $x = 3y$.

$$\iint_R 2xy\,dA = \int_{y=0}^{y=2} \int_{x=y}^{x=3y} 2xy\,dx\,dy.$$

$$= \int_{y=0}^{y=2} x^2 y|_{x=y}^{x=3y}\,dy$$

$$= \int_{y=0}^{y=2}(9y^3 - y^3)dy = \int_{y=0}^{y=2} 8y^3\,dy = 2y^4|_{y=0}^{y=2} = 32\ .$$

When integrating over a disk, or an annulus, or a portion of a disk or an annulus, it is often useful to change to polar coordinates:
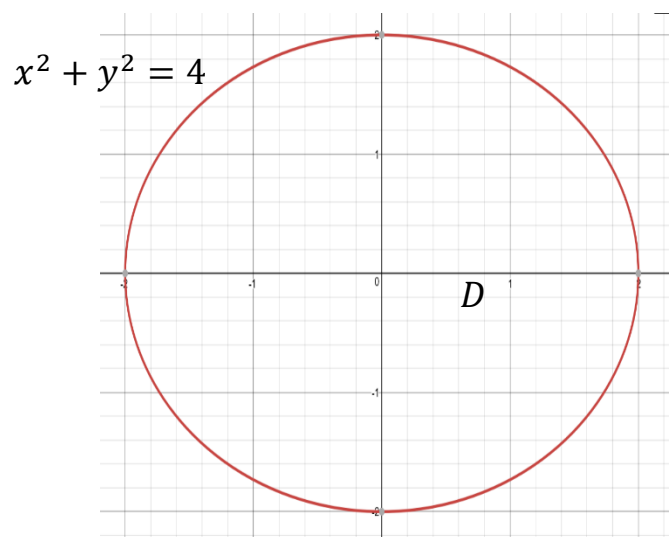
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

$$dA = r\,dr\,d\theta$$

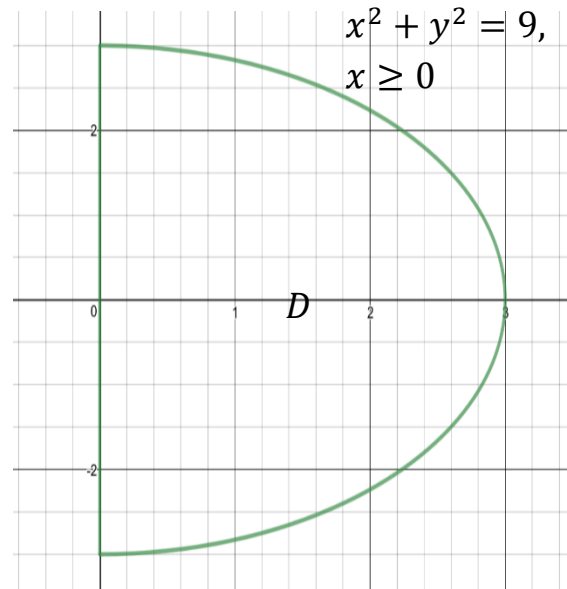Ex. Evaluate $\iint_D (x^2 + y^2)^2 dy dx;$ where $D = \{(x,y)|\ x^2 + y^2 \leq 4\}$

First sketch the region D:

$x^2 + y^2 = 4$

D

$$\iint_D (x^2 + y^2)^2 dy dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (r^2)^2 r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r^5 dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{r^6}{6}|_{r=0}^{r=2} d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1}{6}(2^6 - 0^6) d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{32}{3} d\theta = \frac{32}{3}\theta|_0^{2\pi} = \frac{64\pi}{3}.$$

Ex. Evaluate $\iint_D xy\,dA$, where D is the set where $x^2 + y^2 \leq 9$ and $x \geq 0$.

First sketch the region D:

$x^2 + y^2 = 9,$
$x \geq 0$

$$\iint_D xy\,dA = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=3} (r\cos\theta)(r\sin\theta)r\,dr\,d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=3} r^3\cos\theta\sin\theta\,dr\,d\theta$$

Since the endpoints of integration are constants and $f(r,\theta) = g(r)h(\theta)$ we have

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \cos\theta\sin\theta\,d\theta \int_{r=0}^{r=3} r^3\,dr$$

Let $u = \sin\theta$;   when $\theta = -\frac{\pi}{2}$, $u = -1$
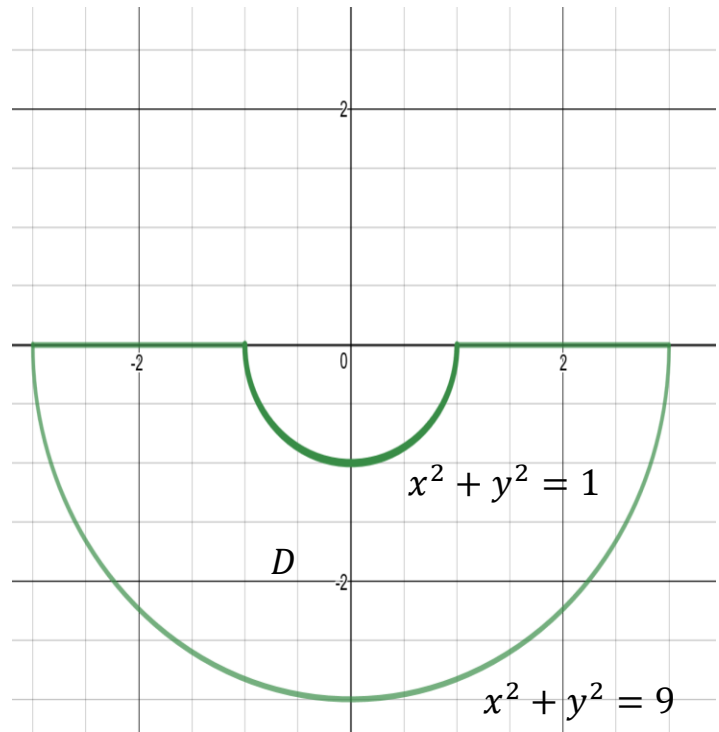
$du = \cos\theta\,d\theta$ when $\theta = \frac{\pi}{2}$, $u = 1$

$$\iint_D xy\,dA = \left(\int_{-1}^{1} u\,du\right)\left(\frac{1}{4}r^4\big|_{r=0}^{r=4}\right)$$

$$= \left(\frac{1}{2}u^2\big|_{u=-1}^{u=1}\right)\left(\frac{1}{4}(4^4)\right) = (0)(4^3) = 0.$$

Ex. Evaluate $\iint_D x^2 dy dx,$ where $D=\{(x,y)|\ 1 \le x^2 + y^2 \le 9,\ and\ y \le 0\}$

First sketch the region D.

$x^2 + y^2 = 1$

D

$x^2 + y^2 = 9$

$$\iint_D x^2 dy dx = \int_{\theta=\pi}^{\theta=2\pi} \int_{r=1}^{r=3} (r^2 \cos^2 \theta) r\, dr\, d\theta$$

$$= \int_{\theta=\pi}^{\theta=2\pi} \int_{r=1}^{r=3} (r^3 \cos^2 \theta) dr\, d\theta$$

Since the endpoints of integration are constants and $f(r,\theta) = g(r)h(\theta)$ we have

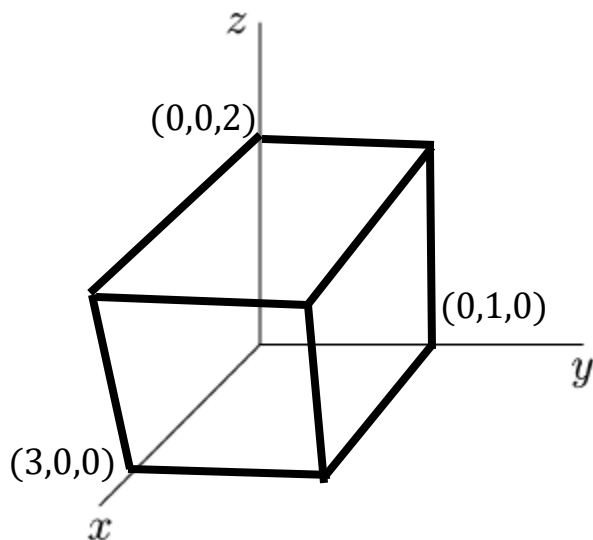$$\iint_D x^2 dy dx = (\int_{\theta=\pi}^{\theta=2\pi} \cos^2 \theta d\theta)(\int_{r=1}^{r=3} r^3 dr)$$

$$= (\int_{\theta=\pi}^{\theta=2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta)(\frac{1}{4} r^4 \ |_{r=1}^{r=3})$$

$$= [\left(\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta\right) |_{\theta=\pi}^{\theta=2\pi}](\frac{1}{4}(3^4 - 1^4))$$

$$= \left[(\pi + 0) - \left(\frac{\pi}{2} + 0\right)\right]\left(\frac{80}{4}\right) = 10\pi .$$

Triple Integrals

If the x, y, and z endpoints of integration are all constants, then you are integrating over a rectangular solid whose sides are parallel to the coordinate planes.

Ex. Evaluate $\iiint_W xy\,dW$, if $W = \{(x, y, z)| 0 \leq x \leq 3, \ 0 \leq y \leq 1, \ 0 \leq z \leq 2\}$
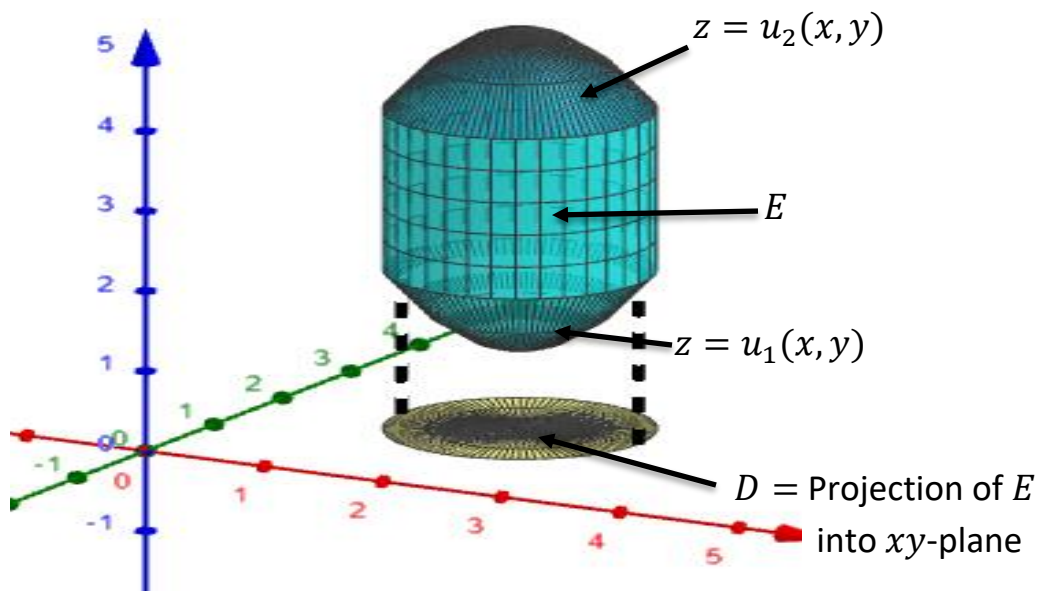
First sketch W:

$$\int_{x=0}^{x=3} \int_{y=0}^{y=1} \int_{z=0}^{z=2} (xy)\,dz\,dy\,dx = \int_{x=0}^{x=3} \int_{y=0}^{y=1} xyz|_{z=0}^{z=2}\,dy\,dx$$

$$= \int_{x=0}^{x=3} \int_{y=0}^{y=1} xy(2 - 0)\,dy\,dx$$

$$= \int_{x=0}^{x=3} \int_{y=0}^{y=1} 2xy\,dy\,dx$$

$$= \int_{x=0}^{x=3} xy^2|_{y=0}^{y=1}\,dx$$

$$= \int_{x=0}^{x=3} x(1^2 - 0^2)\,dx = \int_{x=0}^{x=3} x\,dx = \frac{9}{2}.$$

If E is bounded above by the surface $z = u_2(x, y)$ and below by the surface

$z = u_1(x, y)$, and the projection of the solid E into the xy-plane is D then:

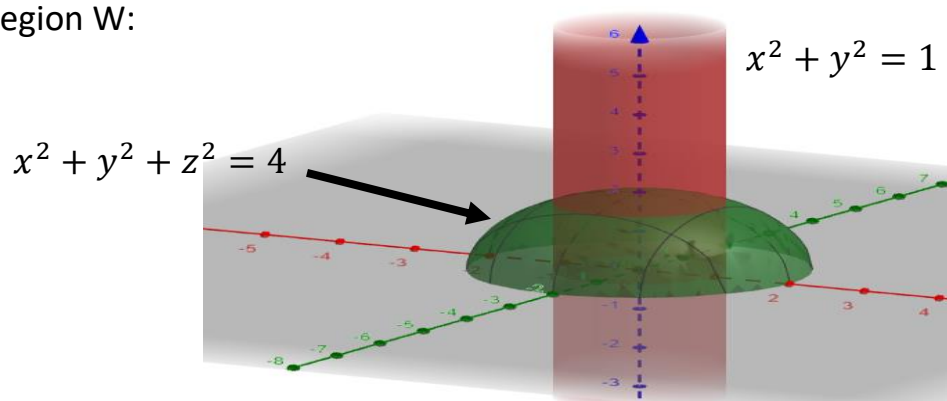$$\iiint_E f(x, y, z)dE = \iint_D \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x, y, z)dzdydx$$

$z = u_2(x, y)$

E

$z = u_1(x, y)$

$D = $ Projection of E
into xy-plane

Note: **If $f(x, y, z) = 1$ then $\iiint_E f(x, y, z)dE = \iiint_E 1\,dE = $ Volume of E.**

Ex. Evaluate $\iiint_W x^2 dW$, where

$W = \{(x, y, z)|\ x^2 + y^2 \le 1,\ z \ge 0,\ x^2 + y^2 + z^2 \le 4\}$

First sketch the region W:

$x^2 + y^2 = 1$

$x^2 + y^2 + z^2 = 4$

$$\iiint_W x^2 dW = \iint_{x^2+y^2\leq 1} \int_{z=0}^{z=\sqrt{4-x^2-y^2}} (x^2) dz dy dx$$

$$= \iint_{x^2+y^2\leq 1} x^2 z|_{z=0}^{z=\sqrt{4-x^2-y^2}} dy dx$$

$$= \iint_{x^2+y^2\leq 1} x^2 (\sqrt{4-x^2-y^2} - 0) dy dx$$

$$= \iint_{x^2+y^2\leq 1} x^2 (\sqrt{4-x^2-y^2}) dy dx.$$

Now change to polar coordinates since we are integrating over a disk

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dy dx = r dr d\theta$$

$$\iiint_W x^2 dW = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (r^2 \cos^2\theta)(\sqrt{4 - r^2\cos^2\theta - r^2\sin^2\theta}) r dr d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (r^2 \cos^2\theta)(\sqrt{4 - r^2}) r dr d\theta.$$

Since the endpoints of integration are constants and $f(r,\theta) = g(r)h(\theta)$ we have

$$\iiint_W x^2 dW = (\int_{\theta=0}^{\theta=2\pi} \cos^2\theta d\theta)(\int_{r=0}^{r=1} r^3\sqrt{4-r^2}\, dr)$$

Let's evaluate each integral separately.

$$\int_{\theta=0}^{\theta=2\pi} \cos^2\theta d\theta = \int_{\theta=0}^{\theta=2\pi} (\frac{1}{2} + \frac{1}{2}\cos 2\theta) d\theta$$

$$= \left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) |_{\theta=0}^{\theta=2\pi} = \pi.$$

$$\int_{r=0}^{r=1} r^3\sqrt{4-r^2}\,dr = \int_{r=0}^{r=1} r^2\sqrt{4-r^2}\,rdr$$

Let $u = 4 - r^2$ which means that $r^2 = 4 - u$.   when $r = 0,\quad u = 4$

$$du = -2rdr \qquad\qquad\qquad\qquad \text{when } r = 1,\quad u = 3.$$

$$-\frac{1}{2}du = rdr$$

$$\int_{r=0}^{r=1} r^2\sqrt{4-r^2}\,rdr = \int_{u=4}^{u=3}(4-u)u^{\frac{1}{2}}\left(-\frac{1}{2}\right)du$$

$$= -\frac{1}{2}\int_{u=4}^{u=3}\left(4u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$$= -\frac{1}{2}\left[\frac{8}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]\Big|_{u=4}^{u=3}$$

$$= \frac{64}{15} - \frac{11}{5}\sqrt{3}\ .$$

Thus we have:

$$\iiint_W x^2\,dW = \left(\int_{\theta=0}^{\theta=2\pi}\cos^2\theta\,d\theta\right)\left(\int_{r=0}^{r=1} r^3\sqrt{1-r^2}\,dr\right) = (\pi)\left(\frac{64}{15} - \frac{11}{5}\sqrt{3}\right)\ .$$