

Stokes' Theorem, the Divergence Theorem, and the Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is:

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Let ω be a differentiable 0-form, i.e., a differentiable function $\omega = f(t)$, then $d\omega = f'(t)dt$.

Let $I = [a, b]$ be the interval $a \leq t \leq b$. Then we define the boundary of I ,

$\partial I = \{b\} - \{a\}$ (this is a formal difference of points, we don't subtract them as real number).

If we then define the zero dimensional integral of a function, f , over a point p to be $f(p)$ the Fundamental Theorem of Calculus becomes:

$$\int_a^b f'(t)dt = \int_I d\omega = \int_{\partial I} \omega = f(b) - f(a).$$

Stokes' Theorem

Stokes' Theorem: Let S be an oriented surface in $\mathbb{R}^3$ with a boundary consisting of a simple closed curve, ∂S , oriented as the boundary of S . Suppose that ω is a

1-form on some open set $K \subseteq \mathbb{R}^3$ that contains S then:

$$\int_{\partial S} \omega = \iint_S d\omega.$$

Proof. Stokes' theorem said $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{S}$.

We can write $\vec{F}$ as: $\vec{F}(x, y, z) = F_1(x, y, z)\vec{i} + F_2(x, y, z)\vec{j} + F_3(x, y, z)\vec{k}$.

Let's let $\omega = \vec{F} \cdot d\vec{S} = F_1 dx + F_2 dy + F_3 dz$ and show that $d\omega = (\nabla \times \vec{F}) \cdot d\vec{S}$.

Then we have:

$$\begin{aligned}
 d\omega &= d(F_1 dx + F_2 dy + F_3 dz) \\
 &= dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz \\
 &= \left(\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz \right) \wedge dx + \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \wedge dy \\
 &\quad + \left(\frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz \right) \wedge dz \\
 &= \frac{\partial F_1}{\partial y} dy \wedge dx + \frac{\partial F_1}{\partial z} dz \wedge dx + \frac{\partial F_2}{\partial x} dx \wedge dy + \frac{\partial F_2}{\partial z} dz \wedge dy \\
 &\quad + \frac{\partial F_3}{\partial x} dx \wedge dz + \frac{\partial F_3}{\partial y} dy \wedge dz \\
 d\omega &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) dy dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) dz dx.
 \end{aligned}$$

We need to show that $(\nabla \times \vec{F}) \cdot d\vec{S} = d\omega$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k}.$$

To find $d\vec{S}$ we need to parametrize the surface S :

$$\vec{\Phi}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$

$$d\vec{S} = (\vec{T}_u \times \vec{T}_v) du dv$$

$$\vec{T}_u = \frac{\partial \vec{\Phi}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad \vec{T}_v = \frac{\partial \vec{\Phi}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

$$\begin{aligned} \vec{T}_u \times \vec{T}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k} \\ &= \frac{\partial(y,z)}{\partial(u,v)} \vec{i} + \frac{\partial(z,x)}{\partial(u,v)} \vec{j} + \frac{\partial(x,y)}{\partial(u,v)} \vec{k}; \end{aligned}$$

thus we have:

$$d\vec{S} = (\vec{T}_u \times \vec{T}_v) du dv = \left(\frac{\partial(y,z)}{\partial(u,v)} \vec{i} + \frac{\partial(z,x)}{\partial(u,v)} \vec{j} + \frac{\partial(x,y)}{\partial(u,v)} \vec{k} \right) du dv.$$

So now we can calculate $(\nabla \times \vec{F}) \cdot d\vec{S}$:

$$\begin{aligned} (\nabla \times \vec{F}) \cdot d\vec{S} &= \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k} \right] \\ &\quad \cdot \left[\frac{\partial(y,z)}{\partial(u,v)} \vec{i} + \frac{\partial(z,x)}{\partial(u,v)} \vec{j} + \frac{\partial(x,y)}{\partial(u,v)} \vec{k} \right] du dv \\ &= \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \frac{\partial(y,z)}{\partial(u,v)} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \frac{\partial(z,x)}{\partial(u,v)} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \frac{\partial(x,y)}{\partial(u,v)} \right] du dv. \end{aligned}$$

Notice that if $x = x(u, v)$, $y = y(u, v)$ then:

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \quad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \quad \text{and}$$

$$\begin{aligned} dx \wedge dy &= \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \wedge \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right) \\ &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} du \wedge dv + \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} dv \wedge du \\ &= \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv = \frac{\partial(x,y)}{\partial(u,v)} du \wedge dv. \end{aligned}$$

Similarly, one can show:

$$dy \wedge dz = \frac{\partial(y,z)}{\partial(u,v)} du \wedge dv \quad \text{and} \quad dz \wedge dx = \frac{\partial(z,x)}{\partial(u,v)} du \wedge dv.$$

Thus our expression for $(\nabla \times \vec{F}) \cdot d\vec{S}$ becomes:

$$\begin{aligned} (\nabla \times \vec{F}) \cdot d\vec{S} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \wedge dy \\ &= d\omega. \end{aligned}$$

So we have:

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_S d\omega \quad \text{and} \quad \int_{\partial S} \vec{F} \cdot d\vec{S} = \int_{\partial S} \omega.$$

Hence we have:

$$\iint_S d\omega = \int_{\partial S} \omega.$$

Ex. Let $\omega = -yx^2dx + xy^2dy$, and let S be the portion of the cone given by

$z^2 = x^2 + y^2$, $0 \leq z \leq 1$. Evaluate $\int_{\partial S} \omega$ both directly and by Stokes' theorem.

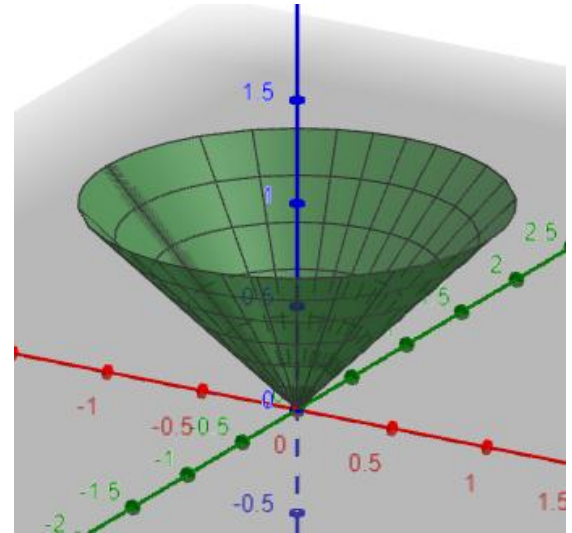
Evaluating $\int_{\partial S} \omega$ directly:

∂S is the unit circle in the plane $z = 1$.

We can parametrize this by: $\vec{c}(t) = \langle \cos t, \sin t, 1 \rangle$,

$0 \leq t \leq 2\pi$.

We then have:



$$dx = \frac{dx}{dt} dt = (-\sin t) dt, \quad dy = \frac{dy}{dt} dt = (\cos t) dt$$

$$\begin{aligned} \int_{\partial S} \omega &= \int_{\partial S} -yx^2 dx + xy^2 dy \\ &= \int_0^{2\pi} -(\sin t)(\cos^2 t)(-\sin t) dt + (\cos t)(\sin^2 t)(\cos t) dt \\ &= \int_0^{2\pi} (\sin^2 t)(\cos^2 t) + (\cos^2 t)(\sin^2 t) dt = \int_0^{2\pi} 2(\sin^2 t)(\cos^2 t) dt \\ &= \int_0^{2\pi} 2\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt \\ &= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos^2 2t\right) dt \\ &= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\cos 4t\right)\right) dt \\ &= \int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{4}\cos 4t\right) dt \\ &= \left(\frac{1}{4}t - \frac{1}{16}\sin 4t\right) \Big|_0^{2\pi} = \frac{\pi}{2}. \end{aligned}$$

Now evaluate by Stokes' theorem: $\iint_S d\omega = \int_{\partial S} \omega$.

$$d\omega = d(-yx^2 dx + xy^2 dy)$$

$$= d(-yx^2) \wedge dx + (-1)^0(-yx^2) \wedge d(dx) + d(xy^2) \wedge dy \\ + (-1)^0(xy^2) \wedge d(dy)$$

$$= (-2xy dx - x^2 dy) \wedge dx + (y^2 dx + 2xy dy) \wedge dy$$

$$= -x^2 dy \wedge dx + y^2 dx \wedge dy = (x^2 + y^2) dx \wedge dy.$$

$$\iint_S d\omega = \iint_S (x^2 + y^2) dx \wedge dy.$$

To evaluate this integral we need to parametrize the cone S .

$$\vec{\Phi}(r, \theta) = \langle r \cos \theta, r \sin \theta, r \rangle; \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

$$dx dy = \frac{\partial(x, y)}{\partial(r, \theta)} dr d\theta = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr d\theta \\ = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} dr d\theta = r dr d\theta.$$

$$\iint_S d\omega = \iint_S (x^2 + y^2) dx \wedge dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^1 (r^2) r dr d\theta = \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^1 r^3 dr = \frac{\pi}{2}.$$

The Divergence Theorem

Theorem (Divergence Theorem): Let $W \subset \mathbb{R}^3$ be an elementary region with ∂W given the outward orientation. If η is a 2-form on some region K which contains W , then

$$\iiint_W d\eta = \iint_{\partial W} \eta.$$

Proof: The Divergence theorem said: $\iiint_W \text{Div } \vec{F} dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}.$

When proving Stokes' theorem we saw that:

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)} du \wedge dv$$

$$dy \wedge dz = \frac{\partial(y,z)}{\partial(u,v)} du \wedge dv$$

$$dz \wedge dx = \frac{\partial(z,x)}{\partial(u,v)} du \wedge dv$$

$$\begin{aligned} \text{and that } d\vec{S} &= (\vec{T}_u \times \vec{T}_v) du dv = \left(\frac{\partial(y,z)}{\partial(u,v)} \vec{i} + \frac{\partial(z,x)}{\partial(u,v)} \vec{j} + \frac{\partial(x,y)}{\partial(u,v)} \vec{k} \right) du dv \\ &= (dy \wedge dz) \vec{i} + (dz \wedge dx) \vec{j} + (dx \wedge dy) \vec{k}. \end{aligned}$$

$$\begin{aligned} \text{So, } \vec{F} \cdot d\vec{S} &= \langle F_1, F_2, F_3 \rangle \cdot \langle (dy \wedge dz), (dz \wedge dx), (dx \wedge dy) \rangle \\ &= F_1 dydz + F_2 dzdx + F_3 dxdy; \end{aligned}$$

Now let's let η be:

$$\eta = F_1 dydz + F_2 dzdx + F_3 dxdy.$$

We need to show that $d\eta = \text{Div } \vec{F} dxdydz.$

$$\begin{aligned}
d\eta &= d(F_1 dydz + F_2 dzdx + F_3 dxdy) \\
&= dF_1 \wedge dydz + (-1)^0(F_1 \wedge d(dydz)) + dF_2 \wedge dzdx \\
&\quad + (-1)^0(F_2 \wedge d(dzdx)) + dF_3 \wedge dxdy + (-1)^0(F_3 \wedge d(dxdy)).
\end{aligned}$$

We saw earlier that $d(dydz) = d(dzdx) = d(dxdy) = 0$. So we have:

$$d\eta = dF_1 \wedge dydz + dF_2 \wedge dzdx + dF_3 \wedge dxdy$$

where:

$$dF_1 \wedge dydz = \left(\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz \right) \wedge dydz = \frac{\partial F_1}{\partial x} dxdydz$$

$$\begin{aligned}
dF_2 \wedge dzdx &= \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \right) \wedge dzdx = \frac{\partial F_2}{\partial y} dydzdx \\
&= (-1)^{(2)(1)} \frac{\partial F_2}{\partial y} dxdydz = \frac{\partial F_2}{\partial y} dxdydz
\end{aligned}$$

$$\begin{aligned}
dF_3 \wedge dxdy &= \left(\frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz \right) \wedge dxdy = \frac{\partial F_3}{\partial z} dzdxdy \\
&= (-1)^{(2)(1)} \frac{\partial F_3}{\partial z} dxdydz = \frac{\partial F_3}{\partial z} dxdydz.
\end{aligned}$$

So now we can say:

$$d\eta = \frac{\partial F_1}{\partial x} dx dy dz + \frac{\partial F_2}{\partial y} dx dy dz + \frac{\partial F_3}{\partial z} dx dy dz$$

$$d\eta = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx dy dz = \text{Div} \vec{F} dx dy dz.$$

Thus we have: $\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iint_{\partial W} \eta$ and $\iiint_W \text{Div} \vec{F} dV = \iiint_W d\eta$.

So by the Divergence theorem: $\iiint_W \text{Div} \vec{F} dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}$, we can conclude:

$$\iiint_W d\eta = \iint_{\partial W} \eta.$$

Ex. Evaluate $\iint_S \eta$, where $\eta = x dy dz$ and S is the unit sphere, directly and by the Divergence Theorem.

Divergence theorem: $\iiint_W d\eta = \iint_{\partial W} \eta$ where W is the unit ball and $\partial W = S$

i.e., $W = \{(x, y, z): x^2 + y^2 + z^2 \leq 1\}$, $S = \{(x, y, z): x^2 + y^2 + z^2 = 1\}$.

$$\eta = x dy dz$$

$$d\eta = d(x dy dz) = dx \wedge (dy dz) + (-1)^0 x \wedge d(dy dz)$$

$$= dx dy dz$$

$$\iiint_W d\eta = \iiint_W dx dy dz = \text{volume}(W) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi.$$

Direct calculation: $\iint_S \eta = \iint_S x dy dz.$

Use the standard parametrization of the unit sphere:

$$\vec{\Phi}(u, v) = \langle \cos v \sin u, \sin v \sin u, \cos u \rangle, \quad 0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi.$$

$$\iint_S x dy dz = \int_{v=0}^{v=2\pi} \int_{u=0}^{u=\pi} (\cos v)(\sin u) \left(\frac{\partial(y,z)}{\partial(u,v)} \right) du dv$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos u (\sin v) & \cos v (\sin u) \\ -\sin u & 0 \end{vmatrix} = \sin^2 u (\cos v)$$

$$\begin{aligned} \iint_S x dy dz &= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=\pi} (\cos v)(\sin u) (\sin^2 u)(\cos v) du dv \\ &= \int_0^{2\pi} \int_0^\pi \cos^2 v (\sin^3 u) du dv = \int_0^{2\pi} (\cos^2 v) dv \int_0^\pi (\sin^3 u) du \\ &= \int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2v}{2} \right) dv \int_0^\pi (\sin^2 u)(\sin u) du \\ &= \left[\left(\frac{1}{2} v + \frac{\sin 2v}{4} \right) \right]_0^{2\pi} \int_0^\pi (1 - \cos^2 u)(\sin u) du \\ &= (\pi) \left(-\cos u + \frac{\cos^3 u}{3} \right) \Big|_0^\pi = (\pi) \left(\frac{4}{3} \right) = \frac{4}{3} \pi. \end{aligned}$$

General Stokes' Theorem: Let M be a (compact) oriented n -dimensional manifold with boundary and ω is a $(n-1)$ -form on M then:

$$\int_M d\omega = \int_{\partial M} \omega.$$