Continuity of Measure

Def. The restriction of the set function outer measure to the class of measurable sets is called the **Lebesgue measure**. We will denote this by m. Thus if E is measurable $m(E) = m^*(E)$.

Prop. Lebesgue measure is countably additive, that is, if $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of sets then $\bigcup_{k=1}^{\infty} E_k$ is measurable and

$$m(\textstyle\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

Proof: We already know that $\bigcup_{k=1}^{\infty} E_k$ is measurable and the outer measure is subadditive thus:

$$m(\textstyle\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k).$$

Now let's prove the inequality in the other direction.

We know for a finite number of disjoint measurable sets:

$$m(\textstyle\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m(E_k).$$

Since $\bigcup_{k=1}^{\infty} E_k \supseteq \bigcup_{k=1}^{n} E_k$ for all n, we have :

$$m(\textstyle\bigcup_{k=1}^{\infty} E_k) \geq m(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m(E_k).$$

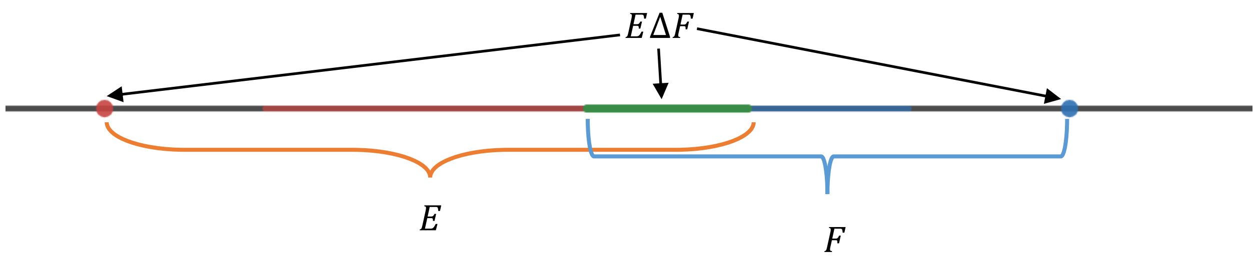
Thus we have: $m(\bigcup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} m(E_k).$

Hence: $m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$

The Lebesgue measure defined on the σ-algebra of Lebesgue measurable sets satisfies:

1. $m(I) = l(I)$
2. $m(t + E) = m(E)$
3. $m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$; E_k are disjoint measurable sets.

Ex. Define $E\Delta F = (E{\sim}F) \cup (F{\sim}E)$. Suppose E and F are measurable sets. Prove $m(E\Delta F) = m(E \cap F^c) + m(F \cap E^c)$.



$$E\Delta F = (E{\sim}F) \cup (F{\sim}E) = (E \cap F^c) \cup (F \cap E^c).$$

Since $(E \cap F^c)$ and $(F \cap E^c)$ are disjoint and measurable we have:

$$m(E\Delta F) = m\big((E \cap F^c) \cup (F \cap E^c)\big) = m\big((E \cap F^c)\big) + m((F \cap E^c)) .$$

Def. $\{E_k\}_{k=1}^{\infty}$ is said to be **ascending** if for each k $E_k \subseteq E_{k+1}$, and **descending** if for each k $E_k \supseteq E_{k+1}$.

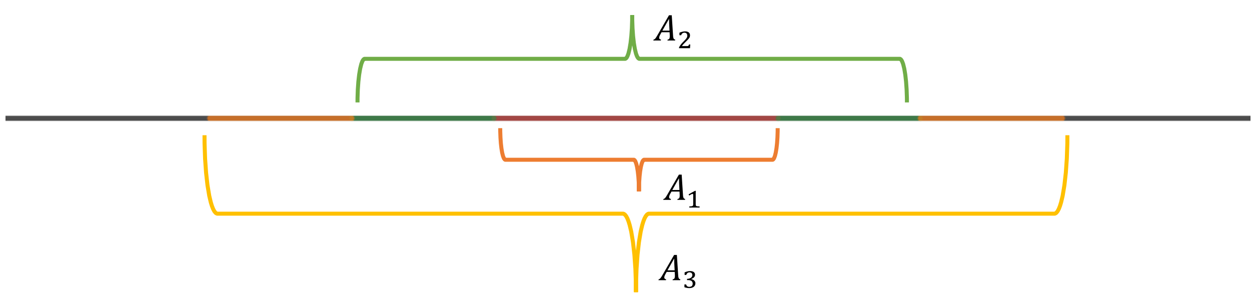
Theorem (The Continuity of Measure):

1. If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then
$$m(\cup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} m(A_k)$$
2. If $\{B_k\}_{k=1}^{\infty}$ is an descending collection of measurable sets and $m(B_1) < \infty$, then
$$m(\cap_{k=1}^{\infty} B_k) = \lim_{k\to\infty} m(B_k).$$

Proof: 1. If for any k, $m(A_k) = \infty$ then by monotonicity

$m(\cup_{k=1}^{\infty} A_k) \geq m(A_k) = \infty$, so the conclusion holds.

If $m(A_k) < \infty$ for all $k \geq 1$, then define $A_0 = \phi$ and $E_k = A_k \sim A_{k-1}$.



Since $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, the E_k's disjoint and $\cup_{k=1}^{\infty} A_k = \cup_{k=1}^{\infty} E_k$:

$$m(\cup_{k=1}^{\infty} A_k) = m(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(A_k \sim A_{k-1}).$$

Since $A_{k-1} \subseteq A_k$:

$$\sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) = \sum_{k=1}^{\infty} (m(A_k) - m(A_{k-1}))$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} (m(A_k) - m(A_{k-1}))$$
$$= \lim_{n\to\infty} m(A_n).$$

To prove 2, let $F_k = B_1 \sim B_k$, for each k.

Since $\{B_k\}_{k=1}^{\infty}$ is descending $\{F_k\}_{k=1}^{\infty}$ is ascending.

By part 1, $m(\bigcup_{k=1}^{\infty} F_k) = \lim_{k\to\infty} m(F_k)$.

However, $\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty}(B_1 \sim B_k) = B_1 \sim \bigcap_{k=1}^{\infty} B_k$.

For each k, $m(B_k) < \infty$ so $m(F_k) = m(B_1) - m(B_k)$. Thus
$\lim_{k\to\infty} m(F_k) = m(\bigcup_{k=1}^{\infty} F_k) = m(B_1 \sim \bigcap_{k=1}^{\infty} B_k) = \lim_{k\to\infty}(m(B_1) - m(B_k))$.

Since $\bigcap_{k=1}^{\infty} B_k \subseteq B_1$

$$m(B_1 \sim \bigcap_{k=1}^{\infty} B_k) = m(B_1) - m(\bigcap_{k=1}^{\infty} B_k).$$

So we have:

$$m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) = m(B_1) - \lim_{k\to\infty}(B_k)$$

Or: $m(\bigcap_{k=1}^{\infty} B_k) = \lim_{k\to\infty} m(B_k)$.

Note: $m(B_1)$ must be finite since if $B_n = [n, \infty)$, then $m(\bigcap_{n=1}^{\infty} B_n) = 0$, but $\lim_{n\to\infty} m(B_n) = \infty$.

Def. For a measurable set E we say a property holds **almost everywhere** (a.e.) on E provided there is a subset $E_0 \subseteq E$ for which $m(E_0) = 0$ and the property holds for all $x \in (E \sim E_0)$.

Ex. Suppose $f(x) = 1$ if x is irrational and $f(x) = 0$ if x is rational.
We would then say that $f(x) = 1$ almost everywhere (a.e.) on $\mathbb{R}$.

The Borel-Cantelli Lemma: Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.

Proof: By countable subadditivity of m we have:

$$m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k) < \infty.$$

If we let $A_n = \bigcup_{k=n}^{\infty} E_k$ then we have $A_{n+1} \subseteq A_n$.

Thus by the continuity of measure:

$$m(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} m(\bigcup_{k=n}^{\infty} E_k)$$
$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0 \ .$$

Thus almost all $x \in \mathbb{R}$ fail to belong to $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} E_k)$.

Hence x belongs to at most a finite number of the E_k's.

Properties of Lebesgue measure

1. Countable and finite additivity. If $\{E_k\}$ (finite or countable) are disjoint and measurable then
$$m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$$

2. Monotonicity: If $A \subseteq B$, A, B measurable then
$$m(A) \leq m(B)$$

3. Excision: If $A \subseteq B$ and $m(A) < \infty$ then
$$m(B{\sim}A) = m(B) - m(A).$$
If $m(A) = 0$ then $m(B{\sim}A) = m(B)$.

4. Countable Monotonicity: For any collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that covers a measurable set E
$$m(E) \leq \sum_{k=1}^{\infty} m(E_k).$$