

Linear Transformations

Def. Let V and W be vector spaces. We call a function $T: V \rightarrow W$ a **linear transformation** from V to W if for all $u, v \in V$ and $c \in \mathbb{R}$

- a. $T(u + v) = T(u) + T(v)$
- b. $T(cv) = cT(v)$.

Theorem: Let $T: V \rightarrow W$ be a linear transformation from a vector space V to a vector space W . Then for $u, v, u_1, \dots, u_n \in V$, $a, b, a_1, \dots, a_n \in \mathbb{R}$

1. $T(0) = 0$
2. $T(v - u) = T(v) - T(u)$
3. $T(au + bv) = aT(u) + bT(v)$
4. $T(\sum_{i=1}^n a_i u_i) = \sum_{i=1}^n a_i T(u_i)$

Proof:

1. $T(0) = T(2(0)) = 2T(0) \implies T(0) = 0$.
2. $T(v - u) = T(v + (-u)) = T(v) + T(-u)$
 $= T(v) - T(u)$.
3. $T(au + bv) = T(au) + T(bv) = aT(u) + bT(v)$.
4. $T(\sum_{i=1}^n a_i u_i) = T(a_1 u_1 + a_2 u_2 + \dots + a_n u_n)$
 $= T(a_1 u_1) + T(a_2 u_2 + \dots + a_n u_n)$
 $= a_1 T(u_1) + T(a_2 u_2) + T(a_3 u_3 + \dots + a_n u_n)$
 \vdots
 $= a_1 T(u_1) + a_2 T(u_2) + \dots + a_n T(u_n)$
 $= \sum_{i=1}^n a_i T(u_i)$.

Ex. Show that $T: V \rightarrow W$ is a linear transformation if and only if $T(cv + u) = cT(v) + T(u)$ for all $u, v \in V$, $c \in \mathbb{R}$.

Proof: Case #3 of the previous theorem shows if T is linear then

$$T(cv + u) = cT(v) + T(u) \text{ for all } u, v \in V, c \in \mathbb{R}.$$

Now let's show that if $T(cv + u) = cT(v) + T(u)$ for all $u, v \in V$, $c \in \mathbb{R}$ then T is linear.

We must show:

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in V$
2. $T(cv) = cT(v)$ for all $c \in \mathbb{R}$.

1. Since $T(cv + u) = cT(v) + T(u)$ for all $u, v \in V$, $c \in \mathbb{R}$, it's true for $c = 1$.

Thus $T(v + u) = T(v) + T(u)$ for all $u, v \in V$.

2. If we take $u = 0$ then $T(cv + 0) = cT(v) + T(0)$

$$\Rightarrow T(cv) = cT(v) + 0 = cT(v).$$

Ex. Show that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\langle a_1, a_2 \rangle) = \langle a_1 + 2a_2, a_1 \rangle$ is a linear transformation.

By the previous example we just need to show that $T(cv + u) = cT(v) + T(u)$ for all $u, v \in \mathbb{R}^2$, $c \in \mathbb{R}$.

For any $u, v \in \mathbb{R}^2$, we have $u = \langle x_1, y_1 \rangle$, $v = \langle x_2, y_2 \rangle$ and

$$\begin{aligned} T(c \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle) &= T(\langle cx_1 + x_2, cy_1 + y_2 \rangle) \\ &= \langle cx_1 + x_2 + 2(cy_1 + y_2), cx_1 + x_2 \rangle. \end{aligned}$$

$$\begin{aligned} cT(\langle x_1, y_1 \rangle) + T(\langle x_2, y_2 \rangle) &= c(\langle x_1 + 2y_1, x_1 \rangle) + (\langle x_2 + 2y_2, x_2 \rangle) \\ &= \langle cx_1 + 2cy_1 + x_2 + 2y_2, cx_1 + x_2 \rangle \\ &= \langle cx_1 + x_2 + 2(cy_1 + y_2), cx_1 + x_2 \rangle \\ &= T(c \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle). \end{aligned}$$

So $T(cu + v) = cT(u) + T(v)$ for all $u, v \in \mathbb{R}^2$, $c \in \mathbb{R}$ and T is linear.

Ex. Define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\langle a_1, a_2 \rangle) = \langle -a_1, a_2 \rangle$. T is a reflection about the y -axis. Show that T is a linear transformation.

For any $u, v \in \mathbb{R}^2$, we have $u = \langle x_1, y_1 \rangle$, $v = \langle x_2, y_2 \rangle$ and

$$\begin{aligned} T(c \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle) &= T(\langle cx_1 + x_2, cy_1 + y_2 \rangle) \\ &= \langle -(cx_1 + x_2), cy_1 + y_2 \rangle. \end{aligned}$$

$$\begin{aligned} cT(\langle x_1, y_1 \rangle) + T(\langle x_2, y_2 \rangle) &= c(\langle -x_1, y_1 \rangle) + \langle -x_2, y_2 \rangle \\ &= \langle -cx_1 - x_2, y_1 + y_2 \rangle \\ &= \langle -(cx_1 + x_2), cy_1 + y_2 \rangle \\ &= T(c \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle) \end{aligned}$$

So T is linear.

Ex. Show that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\langle a, b \rangle) = \langle a + 3, b \rangle$ is not a linear transformation.

Notice that if $c \neq 1$ then $T(cv) \neq cT(v)$:

$$T(c \langle a, b \rangle) = T(\langle ca, cb \rangle) = \langle ca + 3, cb \rangle$$

$$cT(\langle a, b \rangle) = c(a + 3, b) = (c(a + 3), cb) \neq (ca + 3, cb).$$

It's also true that $T(\langle 0, 0 \rangle) \neq \langle 0, 0 \rangle$ and $T(u + v) \neq T(u) + T(v)$ in general.

Ex. Show that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\langle a, b \rangle) = \langle a, 0 \rangle$ (called a **projection**) is a linear transformation.

If we let $u = \langle a_1, b_1 \rangle$, $v = \langle a_2, b_2 \rangle$ and $c \in \mathbb{R}$ then we have:

$$\begin{aligned} T(cu + v) &= T(c \langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle) \\ &= T(\langle ca_1 + a_2, b_1 + b_2 \rangle) \\ &= \langle ca_1 + a_2, 0 \rangle \end{aligned}$$

$$\begin{aligned} cT(u) + T(v) &= cT(\langle a_1, b_1 \rangle) + T(\langle a_2, b_2 \rangle) \\ &= c \langle a_1, 0 \rangle + \langle a_2, 0 \rangle \\ &= \langle ca_1 + a_2, 0 \rangle \end{aligned}$$

Thus $T(cu + v) = cT(u) + T(v)$ for all $u, v \in \mathbb{R}^2$, $c \in \mathbb{R}$, so T is linear.

Ex. Show that $T: P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$ by $T(f(x)) = f'(x)$ is a linear transformation.

Let $f(x), g(x) \in P_n(\mathbb{R})$ and $c \in \mathbb{R}$.

$$\begin{aligned} T(cf(x) + g(x)) &= (cf(x) + g(x))' \\ &= cf'(x) + g'(x) \\ &= cT(f(x)) + T(g(x)). \end{aligned}$$

So T is a linear transformation.

Ex. Let $V = C[a, b]$, the vector space of continuous real valued function on $[a, b]$. Define $T: V \rightarrow \mathbb{R}$ by $T(f(x)) = \int_a^b f(x)dx$. Show that T is a linear transformation.

Let $f(x), g(x) \in C[a, b]$ and $c \in \mathbb{R}$ then

$$\begin{aligned} T(cf(x) + g(x)) &= \int_a^b (cf(x) + g(x))dx \\ &= c \int_a^b f(x)dx + \int_a^b g(x)dx \\ &= cT(f(x)) + T(g(x)). \end{aligned}$$

So T is a linear transformation.

Ex. Show that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\langle a_1, a_2 \rangle) = \langle a_1 a_2, a_2 \rangle$ is not a linear transformation.

Let $v = \langle a_1, a_2 \rangle$ and $c \in \mathbb{R}$, $c \neq 1$ then

$$\begin{aligned} T(cv) &= T(c \langle a_1, a_2 \rangle) \\ &= T(\langle ca_1, ca_2 \rangle) \\ &= \langle c^2 a_1 a_2, ca_2 \rangle \end{aligned}$$

$$\begin{aligned} cT(\langle a_1, a_2 \rangle) &= c \langle a_1 a_2, a_2 \rangle \\ &= \langle ca_1 a_2, ca_2 \rangle \neq T(cv). \end{aligned}$$

Thus T is not a linear transformation.

In this case it also happens to be true that $T(u + v) \neq T(u) + T(v)$ in general.

Ex: Suppose $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation such that $T(\langle 2, 3 \rangle) = \langle 0, 3, 4 \rangle$ and $T(\langle 3, 2 \rangle) = \langle -1, 2, 3 \rangle$. Find $T(\langle 2, 8 \rangle)$.

Notice that we can write $\langle 2, 8 \rangle$ as a linear combination of $\langle 2, 3 \rangle$ and $\langle 3, 2 \rangle$.

$$\begin{aligned} a \langle 2, 3 \rangle + b \langle 3, 2 \rangle &= \langle 2, 8 \rangle \\ \langle 2a + 3b, 3a + 2b \rangle &= \langle 2, 8 \rangle \\ \Rightarrow \quad 2a + 3b &= 2 \\ 3a + 2b &= 8. \end{aligned}$$

Solving these simultaneous equations we get $a = 4, b = -2$.

Thus we have: $4 \langle 2, 3 \rangle - 2 \langle 3, 2 \rangle = \langle 2, 8 \rangle$.

Hence we have:

$$\begin{aligned} T(\langle 2, 8 \rangle) &= T(4 \langle 2, 3 \rangle - 2 \langle 3, 2 \rangle) \\ &= 4T(\langle 2, 3 \rangle) - 2T(\langle 3, 2 \rangle) \\ &= 4 \langle 0, 3, 4 \rangle - 2 \langle -1, 2, 3 \rangle \\ &= \langle 0, 12, 16 \rangle - \langle -2, 4, 6 \rangle \\ &= \langle 2, 8, 10 \rangle. \end{aligned}$$

In fact, given any $\langle x, y \rangle \in \mathbb{R}^2$ we can find $T(\langle x, y \rangle)$ by writing $\langle x, y \rangle$ as a linear combination of $\langle 2, 3 \rangle$ and $\langle 3, 2 \rangle$. In this case we would need to solve:

$$\begin{aligned} a \langle 2, 3 \rangle + b \langle 3, 2 \rangle &= \langle x, y \rangle \\ \Rightarrow \quad 2a + 3b &= x \\ 3a + 2b &= y \end{aligned}$$

for a and b in terms of x and y .

Two important linear transformations are:

1. The identity linear transformation $I: V \rightarrow V$, where $I(v) = v$ for all $v \in V$.
2. The zero linear transformation $T_0: V \rightarrow V$, where $T_0(v) = 0$ for all $v \in V$.

Def. Let V and W be vector spaces, and let $T: V \rightarrow W$ be linear. The **null space** or **kernel** of T , $N(T)$, is the set of $v \in V$ such that $T(v) = 0$. The **range** or **image** of T is the subset of W given by $R(T) = \{T(v) \mid v \in V\}$.

Ex. Let $I: V \rightarrow V$ and $T_0: V \rightarrow W$ be the identity and zero transformations. Then

$$\begin{aligned} N(I) &= \{0\} & N(T_0) &= V \\ R(I) &= V & R(T_0) &= \{0\}. \end{aligned}$$

Ex. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by

$$T(\langle a_1, a_2, a_3 \rangle) = \langle a_1 + a_2, 3a_3 \rangle. \text{ Find } N(T) \text{ and } R(T).$$

To find $N(T)$ we need to find all vectors $\langle a_1, a_2, a_3 \rangle$ such that

$$T(\langle a_1, a_2, a_3 \rangle) = \langle a_1 + a_2, 3a_3 \rangle = \langle 0, 0 \rangle.$$

$$a_1 + a_2 = 0 \quad \Rightarrow \quad a_1 = -a_2$$

$$3a_3 = 0 \quad \Rightarrow \quad a_3 = 0$$

So $N(T) = \{\langle a, -a, 0 \rangle \in \mathbb{R}^3 \mid a \in \mathbb{R}\}$.

$$R(T) = \{T(\langle a_1, a_2, a_3 \rangle) = \langle a_1 + a_2, 3a_3 \rangle \in \mathbb{R}^2 \mid a_1, a_2, a_3 \in \mathbb{R}\}.$$

Let $\langle x, y \rangle$ be any vector in \mathbb{R}^2 . Let's show that $\langle x, y \rangle \in R(T)$.

$$\langle a_1 + a_2, 3a_3 \rangle = \langle x, y \rangle.$$

$$\text{So } a_1 + a_2 = x$$

$$3a_3 = y.$$

In particular if $a_1 = x$, $a_2 = 0$, $a_3 = \frac{y}{3}$ then

$$T\left(\langle x, 0, \frac{y}{3} \rangle\right) = \langle x, y \rangle \implies R(T) = \mathbb{R}^2.$$

Theorem: Let V and W be vector spaces and $T: V \rightarrow W$ be linear. Then $N(T)$ is a subspace of V and $R(T)$ is a subspace of W .

Proof: First we show that $N(T)$ is a subspace of V .

Suppose that $v, w \in N(T)$ and $c \in \mathbb{R}$ then

$$T(v + w) = T(v) + T(w) = 0 + 0 = 0 \implies v + w \in N(T).$$

$$T(cv) = cT(v) = c(0) = 0 \implies cv \in N(T).$$

So $N(T)$ is a subspace of V .

Now we show that $R(T)$ is a subspace of W .

Suppose that $w_1, w_2 \in R(T)$ and $c \in \mathbb{R}$ then there exist $v_1, v_2 \in V$ such that

$$T(v_1) = w_1 \text{ and } T(v_2) = w_2.$$

Thus we have:

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2 \implies w_1 + w_2 \in R(T).$$

$$T(cv_1) = cT(v_1) = cw_1 \implies cw_1 \in R(T).$$

So $R(T)$ is a subspace of W .

Theorem: Let V and W be vector spaces and $T: V \rightarrow W$ be linear. If

$B = \{v_1, \dots, v_n\}$ is a basis for V then

$$R(T) = \text{span}(T(B)) = \text{span}\{T(v_1), \dots, T(v_n)\}.$$

Proof: $T(v_i) \in R(T)$ for each i .

$$R(T) \text{ is a subspace of } W \implies R(T) \text{ contains } \text{span}\{T(v_1), \dots, T(v_n)\}.$$

Now suppose $w \in R(T)$ then $w = T(v)$ for some $v \in V$.

Since $\{v_1, \dots, v_n\}$ is a basis for V , $v = \sum_{i=1}^n a_i v_i$ for some $a_1, \dots, a_n \in \mathbb{R}$.

Since T is linear we have:

$$w = T(v) = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i T(v_i) \in \text{span}(T(B)).$$

So $R(T)$ is contained in the $\text{span}\{T(v_1), \dots, T(v_n)\}$.

Since $R(T) \supseteq \text{span}\{T(v_1), \dots, T(v_n)\}$ and $R(T) \subseteq \text{span}\{T(v_1), \dots, T(v_n)\}$,

$$\implies R(T) = \text{span}(T(B)) = \text{span}\{T(v_1), \dots, T(v_n)\}.$$

Ex. Define the linear transformation $T: P_2(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{bmatrix} f(3) - f(1) & 0 \\ 0 & f(0) \end{bmatrix}.$$

Find a basis for $R(T)$ and $\dim R(T)$.

Since $B = \{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$

$$\begin{aligned} R(T) &= \text{span}\{T(1), T(x), T(x^2)\} \\ &= \text{span}\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}\right\} \\ &= \text{span}\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right\}. \end{aligned}$$

Since $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ are linearly independent (one is not a nonzero multiple of the other) they form a basis for $R(T)$. Hence $\dim R(T) = 2$.

Def. Let V and W be vector spaces and $T: V \rightarrow W$ be linear. If $N(T)$ and $R(T)$ are finite dimensional, then we define the nullity of T , $\mathbf{Nullity}(T) = \mathbf{dim}N(T)$, and the rank of T , $\mathbf{Rank}(T) = \mathbf{dim}R(T)$.

Theorem (dimension theorem): Let V and W be vector spaces and $T: V \rightarrow W$ be linear. If V is a finite dimensional vector space then

$$\text{Nullity}(T) + \text{Rank}(T) = \dim(V).$$

Proof: Suppose that $\dim(V) = n$,

$\dim N(T) = k$ and $\{v_1, \dots, v_k\}$ is a basis for $N(T)$.

Extend $\{v_1, \dots, v_k\}$ to a basis $\{v_1, \dots, v_n\}$ of V .

Claim: $S = \{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $R(T)$.

Since $T(v_1) = T(v_2) = \dots = T(v_k) = 0$, we know from the previous theorem that S generates $R(T)$ since

$$R(T) = \text{span}\{T(v_1), \dots, T(v_n)\} = \text{span}\{T(v_{k+1}), \dots, T(v_n)\}.$$

Now let's show that S is linearly independent. Suppose

$$b_{k+1}T(v_{k+1}) + \dots + b_n T(v_n) = 0.$$

Since T is linear: $T(b_{k+1}(v_{k+1}) + \dots + b_n(v_n)) = 0$.

So $b_{k+1}(v_{k+1}) + \dots + b_n(v_n) \in N(T)$.

Since $\{v_1, \dots, v_k\}$ is a basis for $N(T)$ there exist $c_1, \dots, c_k \in \mathbb{R}$ such that

$$c_1 v_1 + \dots + c_k v_k = b_{k+1}(v_{k+1}) + \dots + b_n(v_n)$$

$$c_1 v_1 + \dots + c_k v_k - b_{k+1}(v_{k+1}) - \dots - b_n(v_n) = 0.$$

But $\{v_1, \dots, v_n\}$ are linearly independent so $b_{k+1}, \dots, b_n = 0$.

Hence S is linearly independent and a basis for $R(T)$.

Thus $\text{Rank}(T) = n - k$ and $\text{Nullity}(T) + \text{Rank}(T) = \dim(V)$.

Ex. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $T(\langle a_1, a_2, a_3 \rangle) = \langle a_1 + a_2, 3a_3 \rangle$. Find the $\dim(R(T))$.

We found earlier that

$$N(T) = \{\langle a, -a, 0 \rangle \mid a \in \mathbb{R}\} = \{a \langle 1, -1, 0 \rangle \mid a \in \mathbb{R}\}.$$

So $N(T)$ has a basis of $\langle 1, -1, 0 \rangle$ and $\dim(N(T)) = 1$.

By the previous theorem we know that $\dim(R(T)) = 2$ since $\dim(\mathbb{R}^3) = 3$.

$$\begin{aligned} \dim(N(T)) + \dim(R(T)) &= \dim(\mathbb{R}^3) \\ 1 + \dim(R(T)) &= 3 \\ \Rightarrow \dim(R(T)) &= 2. \end{aligned}$$

Def. Let V and W be vector spaces and $T: V \rightarrow W$ be linear. T is called

one-to-one if $T(v_1) = T(v_2)$ implies $v_1 = v_2$. T is called **onto** if given any $w \in W$ there exists at least one $v \in V$ such that $T(v) = w$.

Theorem: Let V and W be vector spaces and $T: V \rightarrow W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof: Suppose T is one-to-one and $v \in N(T)$.

$$\text{Then } T(v) = 0 = T(0).$$

But T is one-to-one so $v = 0$.

Now suppose that $N(T) = \{0\}$ and $T(x) = T(y)$.

Then $0 = T(x) - T(y) = T(x - y)$, so $x - y \in N(T)$.

Thus $x - y = 0$ and $x = y$. Thus T is one-to-one.

Theorem: Let V and W be vector spaces of equal (finite) dimension, and let $T: V \rightarrow W$ be linear. Then the following are equivalent:

- a. T is one-to-one
- b. T is onto
- c. $\text{Rank}(T) = \dim(V)$.

Proof: Recall that $\text{Nullity}(T) + \text{Rank}(T) = \dim(V)$.

$$T \text{ is one-to-one} \Leftrightarrow N(T) = \{0\}.$$

$$N(T) = \{0\} \Leftrightarrow \text{Nullity}(T) = 0.$$

$$\text{Nullity}(T) = 0 \Leftrightarrow \text{Rank}(T) = \dim(V).$$

$$\text{Rank}(T) = \dim(V) \Leftrightarrow \text{Rank}(T) = \dim(W).$$

$$\text{Rank}(T) = \dim(W) \Leftrightarrow R(T) = W, \text{ i.e. } T \text{ is onto.}$$

Note: The previous theorem does not hold if V and W are infinite dimensional.

For example, let $V = W = P(\mathbb{R})$ and $T: P(\mathbb{R}) \rightarrow P(\mathbb{R})$ by

$$1. \quad T(f(x)) = \int_0^x f(t) dt.$$

T is one-to-one because $T(f(x)) = T(g(x))$ means

$$\int_0^x f(t) dt = \int_0^x g(t) dt \text{ for all } x.$$

$$\Rightarrow \int_0^x (f(t) - g(t)) dt = 0 \text{ for all } x.$$

$$\Rightarrow f(x) = g(x).$$

However, T is not onto as $T(f(x)) \neq \text{constant function}$.

$$2. T(f(x)) = f'(x).$$

T is not one-to-one because $T(f(x)) = T(g(x))$ means

$$f'(x) = g'(x) \implies f(x) = g(x) + C \text{ for any constant } C.$$

However, T is onto because given any $g(x) = a_0 + a_1x + \dots + a_nx^n \in P(\mathbb{R})$

then $f(x) = \int g(x)dx = a_0x + \frac{1}{2}a_1x^2 + \dots + \frac{a_n}{n}x^{n+1} + C$ has the property that $T(f(x)) = g(x)$.

Ex. Let $T: P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ by $T(f(x)) = xf'(x)$. Show that T is a linear transformation. Determine if T is one-to-one and/or onto.

To show that T is a linear transformation, let $f(x), g(x) \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$.

Then we have:

$$\begin{aligned} T(cf(x) + g(x)) &= x[cf(x) + g(x)]' \\ &= x[cf'(x) + g'(x)] \\ &= cxf'(x) + xg'(x) \\ &= cT(f(x)) + T(g(x)). \end{aligned}$$

Thus T is linear.

T is one-to-one $\iff N(T) = \{0\}$.

$$T(f(x)) = 0$$

$$xf'(x) = 0$$

$$\implies x = 0 \text{ or } f'(x) = 0.$$

But $f'(x) = 0 \implies f(x) = \text{constant}$.

Thus all constant functions $f(x) \in N(T)$.

Hence $N(T) \neq \{0\}$ and T is not one-to-one.

In fact $N(T)$ is spanned by $f(x) = 1$ hence

$$\dim(N(T)) = 1.$$

Since $\text{Nullity}(T) + \text{Rank}(T) = \dim(P_2(\mathbb{R})) = 3$

$\text{Rank}(T) = 2$ so $R(T) \neq P_2(\mathbb{R})$ and T is not onto.

$R(T)$ is spanned by $T(1), T(x), T(x^2)$ because $\{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$.

$$R(T) = \text{span}\{T(1), T(x), T(x^2)\}.$$

$$T(1) = x(0) = 0 \quad \text{since if } f(x) = 1, f'(x) = 0.$$

$$T(x) = x(1) = x \quad \text{since if } f(x) = x, f'(x) = 1.$$

$$T(x^2) = x(2x) = 2x^2 \quad \text{since if } f(x) = x^2, f'(x) = 2x.$$

Thus $R(T) = \text{span}\{0, x, 2x^2\} = \text{span}\{x, 2x^2\}$

$$= \{p(x) \in P_2(\mathbb{R}) \mid p(x) = ax + 2bx^2, a, b \in \mathbb{R}\}$$

Since x and $2x^2$ are linearly independent, $\dim R(T) = 2$.

Ex. Suppose that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear and $T(\langle 0, 1 \rangle) = \langle 2, 3 \rangle$ and $T(\langle 2, -1 \rangle) = \langle 1, 2 \rangle$. Is T one-to-one?

By the previous theorem with $V = W = \mathbb{R}^2$, we have $\dim V = \dim W = 2$.

T is one-to-one if T is onto (i.e. $R(T) = \mathbb{R}^2$).

But since $\langle 0, 1 \rangle$ and $\langle 2, -1 \rangle$ are linearly independent (one is not a multiple of the other), they form a basis for $V = \mathbb{R}^2$. Thus we have:

$$R(T) = \text{span}\{T(\langle 0, 1 \rangle), T(\langle 2, -1 \rangle)\} = \text{span}\{\langle 2, 3 \rangle, \langle 1, 2 \rangle\}.$$

But $\langle 2, 3 \rangle$ and $\langle 1, 2 \rangle$ are also linearly independent and thus a basis for $W = \mathbb{R}^2$.

Hence $R(T) = \mathbb{R}^2$ and T is onto $\Rightarrow T$ is one-to-one

Ex. Let V and W be vector spaces of equal (finite) dimension, and let $T: V \rightarrow W$ be linear. Show that if $\dim(V) > \dim(W)$, then T can't be one-to-one.

$$\text{Nullity}(T) + \text{Rank}(T) = \dim(V) > \dim(W).$$

$R(T)$ is a subspace of W so $\dim(R(T)) \leq \dim(W)$.

Thus $\dim(N(T)) + \text{Rank}(T) > \dim(W) \Rightarrow \dim(N(T)) \geq 1$.

Hence $N(T) \neq \{0\}$ and T is not one-to-one.

Theorem: Let V and W be vector spaces and suppose that $\{v_1, \dots, v_n\}$ is a basis for V . For $w_1, \dots, w_n \in W$, there exists exactly one linear transformation $T: V \rightarrow W$ such that $T(v_i) = w_i$.

Proof: Given any $v \in V$, $v = a_1v_1 + \dots + a_nv_n$, where a_1, \dots, a_n are unique.

Define $T: V \rightarrow W$ by $T(v) = a_1w_1 + \dots + a_nw_n$.

Notice that T is linear since if $u, s \in V$, $d \in \mathbb{R}$ we have:

$u = b_1v_1 + \dots + b_nv_n$ and $s = c_1v_1 + \dots + c_nv_n$ so we get:

$du + s = (db_1 + c_1)v_1 + \dots + (db_n + c_n)v_n$ and

$$\begin{aligned} T(du + s) &= (db_1 + c_1)T(v_1) + \dots + (db_n + c_n)T(v_n) \\ &= (db_1 + c_1)w_1 + \dots + (db_n + c_n)w_n \\ &= d(b_1w_1 + \dots + b_nw_n) + (c_1w_1 + \dots + c_nw_n) \\ &= dT(u) + T(s). \end{aligned}$$

T is unique because if $U: V \rightarrow W$ is a linear transformation with $U(v_i) = w_i$ then

$$\begin{aligned} U(v) &= a_1U(v_1) + \dots + a_nU(v_n) \\ &= a_1w_1 + \dots + a_nw_n \\ &= T(v). \quad (\text{Since } T(v_i) = w_i) \end{aligned}$$

Hence $U = T$.

Corollary: Let V and W be vector spaces and suppose that $\{v_1, \dots, v_n\}$ is a basis for V . If $U, T: V \rightarrow W$ are linear transformations with $U(v_i) = T(v_i)$ for $i = 1, \dots, n$, then $U = T$.

In other words, a linear transformation is defined by what it does to a set of basis vectors.