Orthogonal Projections and the Spectral Theorem

Def. Let V_1 and V_2 be non-empty subsets of a vector space V then:

$$V_1 + V_2 = \{v_1 + v_2 | \ v_1 \in V_1, \ \ v_2 \in V_2\}.$$

Def. A vector space V is called the direct sum of V_1 and V_2, written $V_1 \oplus V_2$, if V_1 and V_2 are subspaces of V such that $V_1 \cap V_2 = \{0\}$ and $V_1 + V_2 = V$.

Ex. If $V = \mathbb{R}^2$, and V_1 is spanned by $< 1,0 >$ and V_2 is spanned by $< 0,1 >$, then $V_1 = x$-axis and $V_2 = y$-axis and $\mathbb{R}^2 = V_1 \oplus V_2$ since

$$V_1 = \{< a, 0 >| \ a \in \mathbb{R}\}, \quad V_2 = \{< 0, b >| \ b \in \mathbb{R}\}$$

and any vector $< a, b >$ in $\mathbb{R}^2$ can be written as $< a, 0 > + < 0, b >$

and $V_1 \cap V_2 = \{0\}$.

Def. Let V_1 and V_2 be subspaces of a vector space V and $V = V_1 \oplus V_2$. $T: V \to V$ is called a **projection on V_1 along V_2** if for any $v = v_1 + v_2$, $v_1 \in V_1, \ v_2 \in V_2$ we have $T(v) = v_1$.

Ex. Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$. Then $T(a, b) = (a, 0)$ is a projection on

$V_1 = x$-axis$= \{< a, 0 >| \ a \in \mathbb{R}\}$, along $V_2 = y$-axis$= \{< 0, b >| \ b \in \mathbb{R}\}$.

Ex. Let $T: V \to V$, with

$$V_1 = R(T) = \{v_1 \in V | \, T(w) = v_1, \text{for some } w \in V\}$$

$$V_2 = N(T) = \{v_2 \in V | \, T(v_2) = 0\}.$$

Then $V = R(T) \oplus N(T)$ and T is a projection on to $V_1 = R(T)$ along $V_2 = N(T)$.

Notice that

$$V = V_1 \oplus V_2 = V_1 \oplus V_3 \text{ does } \textbf{NOT} \text{ imply } V_2 = V_3.$$

For example, let $V = \mathbb{R}^2$,

$$V_1 = \{< a, 0 > | \, a \in \mathbb{R}\}$$

$$V_2 = \{< 0, b > | \, b \in \mathbb{R}\}$$

$$V_3 = \{< c, c > | \, c \in \mathbb{R}\}.$$

Since $< 1,0 >, < 0,1 >$ spans $\mathbb{R}^2$ and $< 1,0 >, < 1,1 >$ spans $\mathbb{R}^2$ every vector in $\mathbb{R}^2$ can be written as $v_1 + v_2, \; v_1 \in V_1, \; v_2 \in V_2$ or $v_1 + v_3, \; v_1 \in V_1, \; v_3 \in V_3$. In addition, $V_1 \cap V_2 = \{0\}$ and $V_1 \cap V_3 = \{0\}$. so:

$$V = V_1 \oplus V_2 = V_1 \oplus V_3, \quad \text{but } V_2 \neq V_3.$$

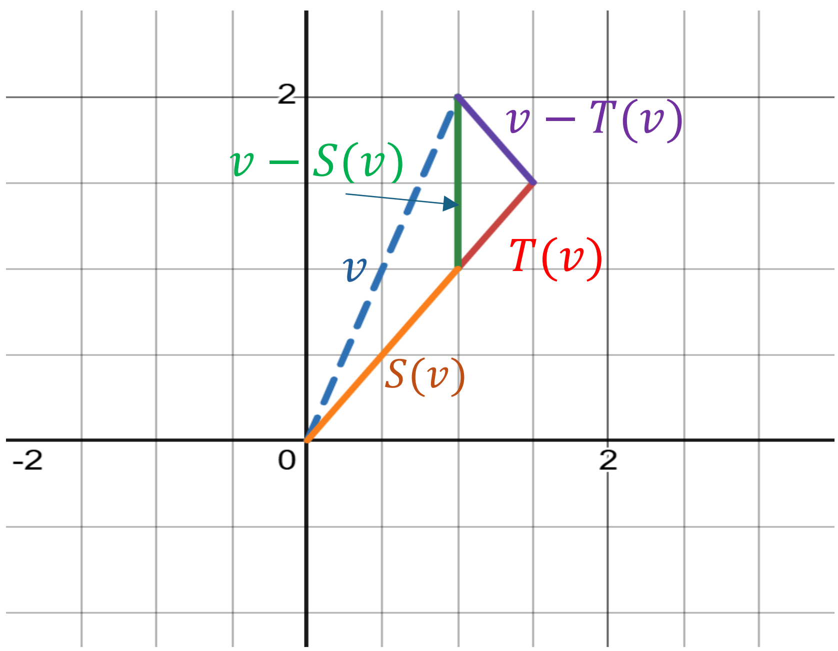
So V_1 does not uniquely determine T.

We can make T unique with the following definition.

Def. Let V be a real inner product space and $T: V \to V$ a projection. We say T is an **orthogonal projection** if $R(T)^{\perp} = N(T)$ (which is equivalent to $N(T)^{\perp} = R(T)$) where $W^{\perp}$ means the set of vectors in V perpendicular to all of the vectors in W (this is also called the perpendicular complement of W).

To see the difference between a projection and an orthogonal projection, let $V_1 = \{< a, a >|\ a \in \mathbb{R}\} \subseteq \mathbb{R}^2$. We consider two projections of $\mathbb{R}^2$ onto V_1.

$$S(a, b) =< a, a > \quad \text{and} \quad T(a, b) =< \frac{a+b}{2}, \frac{a+b}{2} > .$$



S is a "vertical" projection onto V_1 while T is an orthogonal projection onto V_1. That is, in general $v - S(v)$ (green vector) is not perpendicular to V_1, while $v - T(v)$ (purple vector) is perpendicular to V_1.

In general, if we want to find the orthogonal projection of $v =< x, y >$ onto a unit vector $u =< a, b >$, we know from Calculus 3 that:

$$Proj_u v = \left[\frac{<v,u>}{\|u\|^2}\right] u = [< v, u >]u.$$

So if T is a linear transformation mapping any vector $v \in \mathbb{R}^2, v =< x, y >$, onto its orthogonal projection onto a unit vector $u =< a, b >$ we get:

$$T(x, y) = [< x, y > \cdot < a, b >] < a, b >$$
$$= (ax + by) < a, b >$$
$$=< a^2x + aby,\ abx + b^2y >.$$

So with respect to the standard ordered basis β on $\mathbb{R}^2$ we get:

$$T(1,0) =< a^2, ab >$$
$$T(0,1) =< ab, b^2 >$$

$$[T]_\beta = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}.$$

Notice that this is just:

$$uu^t = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}.$$

In particular, if u is an eigenvector of length one, the orthogonal projection onto u can be represented as: $T = uu^t$.

Spectral Theorem: Suppose that T is a linear operator on a real finite dimensional inner product space V with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Assume that T is self-adjoint. For each i, let V_i be the eigenspace of T corresponding to λ_i and let T_i be the orthogonal projection of V onto V_i. Then we have:

a. $V = V_1 \oplus \cdots \oplus V_k$
b. If V_i' is the direct sum of the subspaces $V_j,\ j \neq i$, then $V_i' = V_i^{\perp}$.
c. $T_i T_j = \delta_{ij} T_i; \quad 1 \leq i, j \leq k$.
d. $T_1 + T_2 + \cdots + T_k = I$
e. $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$.

The set of eigenvalues $\{\lambda_1, \dots, \lambda_k\}$ is called the **spectrum of T**,
$I = T_1 + T_2 +,,, + T_k$, is called the **resolution of the identity** and
$T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ is called the **spectral decomposition of T**.

Let β be the union of the orthonormal bases of the eigenspaces V_i, and let $m_i = dimV_i$, then we have:

$$[T]_\beta = \left(\begin{pmatrix} \lambda_1 I_{m1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k I_{mk} \end{pmatrix}\right).$$

To find the spectral decomposition of $A = [T]: V \to V$:

1. Find the eigenvalues of A
2. Find the eigenvectors/eigenspaces of A
3. Find an orthonormal basis of eigenvectors of A, $u_1, \dots, u_k$
4. Find the orthogonal projections T_i of V
5. Write $A = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$, the spectral decomposition of A.

Ex. Find the spectral decomposition of $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$.

In the section called "Normal and Self-Adjoint Operators" we found an orthonormal basis for $\mathbb{R}^2$ of eigenvectors of A (if we hadn't, we would need to first find them for this matrix):

$$u_1 = \frac{1}{\sqrt{5}} < 2,1 >, \qquad \text{associated with } \lambda_1 = 1$$

$$u_2 = \frac{1}{\sqrt{5}} < 1,-2 >, \quad \text{associated with } \lambda_2 = 6.$$

The spectral decomposition of A is:

$$A = \lambda_1[T_1] + \lambda_2[[T_2] ;$$

Where T_1 is the orthogonal projection of $\mathbb{R}^2$ on the eigenspace spanned by $u_1 = \frac{1}{\sqrt{5}} < 2,1 >$ and T_2 is the orthogonal projection of $\mathbb{R}^2$ on the eigenspace spanned by $u_2 = \frac{1}{\sqrt{5}} < 1,-2 >$.

$$T_1 = [u_1][u_1]^t = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$$T_2 = [u_2][u_2]^t = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

Notice that $T_1 + T_2 = I$.

So the spectral decomposition of A is:

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} = 1\left[\frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\right] + 6[\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}].$$

Ex. Find the spectral decomposition of $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.

In the section called "Normal and Self-Adjoint Operators" we found an orthonormal basis for $\mathbb{R}^3$ of eigenvectors of A:

$u_1 = \frac{1}{\sqrt{2}} < -1,1,0 >,$ associated with $\lambda_1 = 2$

$u_2 = \frac{1}{\sqrt{6}} < 1,1,-2 >,$ associated with $\lambda_2 = 2$

$u_3 = \frac{1}{\sqrt{3}} < 1,1,1 >,$ associated with $\lambda_3 = 8.$

We can find the orthogonal projections of $\mathbb{R}^3$ onto each eigenvector by $T_i = u_i(u_i)^t$.

$$[T_1] = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\left(\frac{1}{\sqrt{2}}\right)(-1 \quad 1 \quad 0) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[T_2] = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\left(\frac{1}{\sqrt{6}}\right)(1 \quad 1 \quad -2) = \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}$$

$$[T_3] = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\left(\frac{1}{\sqrt{3}}\right)(1 \quad 1 \quad 1) = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Notice that $T_1 + T_2 + T_3 = I$.

The spectral decomposition of A is:

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} =$$

$$= 2\left[\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] + 2\left[\frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}\right] + 8[\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}].$$