

The Adjoint of a Linear Operator

Let V be an inner product space and T a linear operator on V (i.e., T is a linear transformation from V into V).

Def. The **Adjoint of T** , written T^* , is a linear operator on V whose matrix representation with respect to an orthonormal basis β of V is $[T]^*_{\beta} = A^*$, where A^* is the transpose of $A = [T]_{\beta}$. (Note: if V was a vector space over the complex numbers then A^* would be the conjugate transpose of A . However, we will only be considering vector spaces over the real numbers).

Theorem: Let V be a finite dimensional inner product space over \mathbb{R} and $F: V \rightarrow \mathbb{R}$, a linear transformation. Then there exists a unique $y \in V$ such that $F(v) = \langle v, y \rangle$ for all $v \in V$.

Proof: Let $\beta = \{v_1, \dots, v_n\}$ be an orthonormal basis for V and

$$y = \sum_{i=1}^n F(v_i)v_i.$$

Define a linear transformation $h: V \rightarrow \mathbb{R}$ by $h(v) = \langle v, y \rangle$.

For $1 \leq i \leq n$ we have:

$$\begin{aligned} h(v_j) &= \langle v_j, \sum_{i=1}^n F(v_i)v_i \rangle = \sum_{i=1}^n F(v_i) \langle v_i, v_j \rangle \\ &= F(v_j) \quad (\text{since } \{v_1, \dots, v_n\} \text{ is orthonormal}). \end{aligned}$$

Since F and h are both linear transformations and they are equal on the basis β , they must be equal on all of V .

To see that y is unique, suppose there is a y' such that

$$F(v) = \langle v, y' \rangle \quad \text{for all } v \in V.$$

Then $\langle v, y \rangle = \langle v, y' \rangle$ for all $v \in V$.

But by an earlier theorem $y = y'$, so y is unique.

Ex. Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F(v_1, v_2) = 4v_1 - v_2$. Find the unique vector $y \in \mathbb{R}^2$ such that $F(v_1, v_2) = \langle v, y \rangle$ where $v = (v_1, v_2)$.

Let $\beta = \{e_1, e_2\}$ be the standard ordered orthonormal basis for \mathbb{R}^2 .

$$\begin{aligned} y &= (F(e_1))e_1 + (F(e_2))e_2 \\ &= 4e_1 - 1e_2 = (4, -1). \end{aligned}$$

Then: $F(v_1, v_2) = \langle (v_1, v_2), (4, -1) \rangle = 4v_1 - v_2$.

Theorem: Let V be a finite dimensional inner product space over \mathbb{R}

and T a linear operator on V . Then there exists a unique function $T^*: V \rightarrow V$ such that $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$ for all $v, w \in V$.

Furthermore, T^* is linear.

Proof: Let $w \in V$ and define $G: V \rightarrow \mathbb{R}$ by $G(v) = \langle T(v), w \rangle$, for $v \in V$.

First notice that G is linear since for any $v_1, v_2 \in V$ and $c \in \mathbb{R}$:

$$\begin{aligned} G(cv_1 + v_2) &= \langle T(cv_1 + v_2), w \rangle \\ &= \langle cT(v_1) + T(v_2), w \rangle \quad \text{because } T \text{ is linear} \\ &= c \langle T(v_1), w \rangle + \langle T(v_2), w \rangle \\ &= cG(v_1) + G(v_2). \end{aligned}$$

By the previous theorem there is a unique $w' \in V$ such that:

$$G(v) = \langle v, w' \rangle.$$

Thus we have for all $v \in V$:

$$G(v) = \langle T(v), w \rangle = \langle v, w' \rangle.$$

So we can define: $T^*: V \rightarrow V$ by $T^*(w) = w'$.

Thus we have: $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$.

Let's show that T^* is linear.

Let $w_1, w_2 \in V$ and $c \in \mathbb{R}$, then for any $v \in V$ we have:

$$\begin{aligned} \langle v, T^*(cw_1 + w_2) \rangle &= \langle T(v), cw_1 + w_2 \rangle \\ &= c \langle T(v), w_1 \rangle + \langle T(v), w_2 \rangle \\ &= c \langle v, T^*(w_1) \rangle + \langle v, T^*(w_2) \rangle \\ &= \langle v, cT^*(w_1) + T^*(w_2) \rangle. \end{aligned}$$

Since this is true for any $v \in V$, we have:

$$T^*(cw_1 + w_2) = cT^*(w_1) + T^*(w_2).$$

Thus T^* is linear.

Now let's show that T^* is unique.

Suppose $S: V \rightarrow V$ is linear and for all $v, w \in V$

$$\langle T(v), w \rangle = \langle v, S(w) \rangle.$$

Since $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$ for all $v, w \in V$, we have

$$\langle v, T^*(w) \rangle = \langle v, S(w) \rangle.$$

Thus $T^*(w) = S(w)$ for all $w \in V$.

Hence $T^* = S$, and T^* is unique.

The next theorem tells us that we can calculate a matrix representation of T^* , called that **adjoint of T** , by taking the transpose of a matrix representation of T .

Theorem: Let V be a finite dimensional inner product space over \mathbb{R} and β an orthonormal basis for V . If $T: V \rightarrow V$ is a linear operator on V then

$$[T^*]_{\beta} = [T]_{\beta}^* .$$

Proof: Let $A = [T]_{\beta}$, $B = [T^*]_{\beta}$, where $\beta = \{v_1, \dots, v_n\}$ is an orthonormal basis for V .

$$\begin{aligned} B_{ij} &= \langle T^*(v_j), v_i \rangle = \langle v_i, T^*(v_j) \rangle \\ &= \langle T(v_i), v_j \rangle \\ &= A_{ji} \\ &= (A^*)_{ij}. \end{aligned}$$

So $B = A^*$, where A^* is the transpose of the matrix A .

Ex. Let T be a linear operator on \mathbb{R}^3 given by

$$T(a_1, a_2, a_3) = ((3a_1 - a_3), (a_1 + 2a_2), (a_1 + a_2 + a_3))$$

With respect to β , the standard ordered basis on \mathbb{R}^3 . Find the matrix representation of T^* with respect to this basis.

We start by finding a matrix representation of T with respect to β .

$$[T]_{\beta} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

So T^* is the transpose of this matrix:

$$[T^*]_{\beta} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Ex. Show that in the previous example $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$, for all $v, w \in \mathbb{R}^3$.

Let $v = \langle a_1, a_2, a_3 \rangle$, $w = \langle b_1, b_2, b_3 \rangle$.

$$T(v) = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 3a_1 - a_3 \\ a_1 + 2a_2 \\ a_1 + a_2 + a_3 \end{pmatrix}$$

$$\begin{aligned} \langle T(v), w \rangle &= \langle ((3a_1 - a_3), (a_1 + 2a_2), (a_1 + a_2 + a_3)), (b_1, b_2, b_3) \rangle \\ &= (3a_1 - a_3)b_1 + (a_1 + 2a_2)b_2 + (a_1 + a_2 + a_3)b_3. \end{aligned}$$

$$T^*(w) = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3b_1 + b_2 + b_3 \\ 2b_2 + b_3 \\ -b_1 + b_3 \end{pmatrix}$$

$$\begin{aligned} \langle v, T^*(w) \rangle &= \langle (a_1, a_2, a_3), ((3b_1 + b_2 + b_3), (2b_2 + b_3), (-b_1 + b_3)) \rangle \\ &= a_1(3b_1 + b_2 + b_3) + a_2(2b_2 + b_3) + a_3(-b_1 + b_3) \\ &= (3a_1 - a_3)b_1 + (a_1 + 2a_2)b_2 + (a_1 + a_2 + a_3)b_3 \\ &= \langle T(v), w \rangle. \end{aligned}$$

Theorem: Let V be a real inner product space and S and T linear operators on V . Then

- a. $(S + T)^* = S^* + T^*$
- b. $(cT)^* = cT^*$, $c \in \mathbb{R}$
- c. $(TS)^* = S^*T^*$
- d. $T^{**} = T$
- e. $I^* = I$.

Below are the proofs of a and c .

- a. For $v, w \in V$ we have;

$$\begin{aligned}
 \langle v, (S + T)^*(w) \rangle &= \langle (S + T)(v), w \rangle \\
 &= \langle (S(v) + T(v)), w \rangle \\
 &= \langle S(v), w \rangle + \langle T(v), w \rangle \\
 &= \langle v, S^*(w) \rangle + \langle v, T^*(w) \rangle
 \end{aligned}$$

For all $v, w \in V$ so

$$(S + T)^* = S^* + T^*.$$

- c. For all $v, w \in V$ we have:

$$\begin{aligned}
 \langle v, S^*T^*(w) \rangle &= \langle S(v), T^*(w) \rangle \\
 &= \langle (TS)(v), w \rangle \\
 &= \langle v, (TS)^*(w) \rangle
 \end{aligned}$$

So $(TS)^* = S^*T^*$.

Corollary: Let A and B be $n \times n$ matrices. Then:

- a. $(A + B)^* = A^* + B^*$
- b. $(cA)^* = cA^*$, for any $c \in \mathbb{R}$
- c. $(AB)^* = B^*A^*$
- d. $A^{**} = A$
- e. $I^* = I$.

Ex. Let V be a finite dimensional real inner product space. Prove that if T is invertible then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Since T is invertible, it has an inverse, T^{-1} , and $TT^{-1} = I$.

We can then say that $(TT^{-1})^* = I^*$.

By part e of the last theorem $I^* = I$, and by part c, $(TT^{-1})^* = (T^{-1})^*T^*$.

Thus we have: $(T^{-1})^*T^* = I$.

That means that T^* has an inverse given by $(T^{-1})^*$.

That is, $(T^*)^{-1} = (T^{-1})^*$.