The Adjoint of a Linear Operator

Let V be an inner product space and T a linear operator on V (i.e., T is a linear transformation from V into V).

Def. The **Adjoint of $\boldsymbol{T}$**, written T^*, is a linear operator on V whose matrix representation with respect to an orthonormal basis β of V is $[T]^*{}_\beta = A^*$, where A^* is the transpose of $A = [T]_\beta$. (Note: if V was a vector space over the complex numbers then A^* would be the conjugate transpose of A. However, we will only be considering vector spaces over the real numbers).

Theorem: Let V be a finite dimensional inner product space over $\mathbb{R}$ and $F: V \to \mathbb{R}$, a linear transformation. Then there exists a unique $y \in V$ such that $F(v) =< v, y >$ for all $v \in V$.

Proof: Let $\beta = \{v_1, \dots, v_n\}$ be an orthonormal basis for V and

$$y = \sum_{i=1}^{n} F(v_i) v_i.$$

Define a linear transformation $h: V \to \mathbb{R}$ by $h(v) =< v, y >$, for $1 \leq i \leq n$.

Then we have:

$$h(v_j) =< v_j, \sum_{i=1}^{n} F(v_i) v_i >= \sum_{i=1}^{n} F(v_i) < v_i, v_j >$$

$$= F(v_j) \quad \text{(since } \{v_1, \dots, v_n\} \text{ is orthonormal).}$$

Since F and h are both linear transformations and they are equal on the basis β, they must be equal on all of V.

To see that y is unique, suppose there is a y' such that

$$F(v) =< v, y' > \quad \text{for all } v \in V.$$

Then $< v, y >=< v. y' >$ for all $v \in V$.

But by an earlier theorem $y = y'$, so y is unique.

Ex. Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F(v_1, v_2) = 4v_1 - v_2$. Find the unique vector $y \in \mathbb{R}^2$ such that $F(v_1, v_2) =< v, y >$ where $v = (v_1, v_2)$.

Let $\beta = \{e_1, e_2\}$ be the standard ordered orthonormal basis for $\mathbb{R}^2$.

$$y = \big(F(e_1)\big)e_1 + \big(F(e_2)\big)e_2$$

$$= 4e_1 - 1e_2 = (4, -1).$$

Then: $F(v_1, v_2) =< (v_1, v_2), (4, -1) >= 4v_1 - v_2$.

Theorem: Let V be a finite dimensional inner product space over $\mathbb{R}$

And T a linear operator on V. Then there exists a unique function $T^*: V \rightarrow V$ such that $< T(v), w >=< v, T^*(w) >$ for all $v, w \in V$. Furthermore, T^* is linear.

Proof: Let $w \in V$ and define $G: V \to \mathbb{R}$ by $G(v) =< T(v),\ w >$, for $v \in V$.

First notice that G is linear since for any $v_1, v_2 \in V$ and $c \in \mathbb{R}$:

$$G(cv_1 + v_2) =< T(cv_1 + v_2), w >$$

$$=< cT(v_1) + T(v_2), w > \quad \text{because } T \text{ is linear}$$

$$= c < T(v_1), w > +< T(v_2), w >$$

$$= cG(v_1) + G(v_2).$$

By the previous theorem there is a unique $w' \in V$ such that:

$$G(v) =< v, w' >.$$

Thus we have for all $v \in V$:

$$G(v) =< T(v), w >=< v, w' >.$$

So we can define: $T^*: V \to V$ by $T^*(w) = w'$.

Thus we have: $< T(v), w >=< v, T^*(w) >$.

Let's show that T^* is linear.

Let $w_1, w_2 \in V$ and $c \in \mathbb{R}$, then for any $v \in V$ we have:

$$< v,\ T^*(cw_1 + w_2) >=< T(v),\ cw_1 + w_2 >$$

$$= c < T(v), w_1 > +< T(v), w_2 >$$

$$= c < v, T^*(w_1) > +< v, T^*(w_2) >$$

$$=< v, cT^*(w_1) + T^*(w_2) >.$$

Since this is true for any $v \in V$, we have:

$$T^*(cw_1 + w_2) = cT^*(w_1) + T^*(w_2).$$

Thus T^* is linear.

Now let's show that T^* is unique.

Suppose $S: V \to V$ is linear and for all $v, w \in V$

$$< T(v), w > = < v, S(w) >.$$

Since $< T(v), w > = < v, T^*(w) >$ for all $v, w \in V$, we have

$$< v, T^*(w) > = < v, S(w) >.$$

Thus $\qquad T^*(w) = S(w)$ for all $w \in V$.

Hence $T^* = S$, and T^* is unique.

The next theorem tells us that we can calculate a matrix representation of T^*, called that **adjoint of $\boldsymbol{T}$**, by taking the transpose of a matrix representation of T.

Theorem: Let V be a finite dimensional inner product space over $\mathbb{R}$ and β an orthonormal basis for V. If $T: V \to V$ is a linear operator on V then

$$[T^*]_\beta = [T]_\beta{}^* .$$

Proof: Let $A = [T]_\beta$, $B = [T^*]_\beta$, where $\beta = \{v_1, \dots, v_n\}$ is an orthonormal basis for V.

$$B_{ij} =< T^*(v_j), v_i >=< v_i, T^*(v_j) >$$
$$=< T(v_i), v_j >$$
$$= A_{ji}$$
$$= (A^*)_{ij}.$$

So $B = A^*$, where A^* is the transpose of the matrix A.

Ex. Let T be a linear operator on $\mathbb{R}^3$ given by

$$T(a_1, a_2, a_3) = ((3a_1 - a_3), (a_1 + 2a_2), (a_1 + a_2 + a_3))$$

With respect to β, the standard ordered basis on $\mathbb{R}^3$. Find the matrix representation of T^* with respect to this basis.

We start by finding a matrix representation of T with respect to β.

$$[T]_\beta = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

So T^* is the transpose of this matrix:

$$[T^*]_\beta = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Ex. Show that in the previous example $< T(v)\,, w >=< v,\ T^*(w) >$, for all $v, w \in \mathbb{R}^3$.

Let $v =< a_1, a_2, a_3 >$, $w =< b_1, b_2, b_3 >$.

$$T(v) = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 3a_1 - a_3 \\ a_1 + 2a_2 \\ a_1 + a_2 + a_3 \end{pmatrix}$$

$$< T(v), w >=< ((3a_1 - a_3), (a_1 + 2a_2),\ (a_1 + a_2 + a_3)), (b_1, b_2, b_3) >$$

$$= (3a_1 - a_3)b_1 + (a_1 + 2a_2)b_2 + (a_1 + a_2 + a_3)b_3.$$

$$T^*(w) = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3b_1 + b_2 + b_3 \\ 2b_2 + b_3 \\ -b_1 + b_2 \end{pmatrix}$$

$$< v, T^*(w) >=< (a_1, a_2, a_3), ((3b_1 + b_2 + b_3), (2b_2 + b_3), (-b_1 + b_2)) >$$

$$= a_1(3b_1 + b_2 + b_3) + a_2(2b_2 + b_3) + \ a_3(-b_1 + b_2)$$

$$= (3a_1 - a_3)b_1 + (a_1 + 2a_2)b_2 + (a_1 + a_2 + a_3)b_3$$

$$=< T(v), w >.$$

Theorem: Let V be a real inner product space and S and T linear operators on V. Then

a. $(S+T)^* = S^* + T^*$
b. $(cT)^* = cT^*, \ c \in \mathbb{R}$
c. $(TS)^* = S^*T^*$
d. $T^{**} = T$
e. $I^* = I$.

Below are the proofs of a and c.

a. For $v, w \in V$ we have;

$$< v, (S+T)^*(w) > = < (S+T)(v), w >$$
$$= < \big(S(v) + T(v)\big), w >$$
$$= < S(v), w > + < T(v), w >$$
$$= < v, S^*(w) > + < v, T^*(w) >$$

For all $v, w \in V$ so

$$(S+T)^* = S^* + T^*.$$

c. For all $v, w \in V$ we have:

$$< v, \ S^*T^*(w) > = < S(v), T^*(w) >$$
$$= < (TS)(v), w >$$
$$= < v, \ (TS)^*(w) >$$

So $(TS)^* = S^*T^*$.

Corollary: Let A and B be nxn matrices. Then:

a. $(A+B)^* = A^* + B^*$
b. $(cA)^* = cA^*$, for any $c \in \mathbb{R}$
c. $(AB)^* = B^*A^*$
d. $A^{**} = A$
e. $I^* = I$.

Ex. Let V be a finite dimensional real inner product space. Prove that if T is invertible then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Since T is invertible, it has an inverse, T^{-1}, and $TT^{-1} = I$.

We can then say that $(TT^{-1})^* = I^*$.

By part e of the last theorem $I^* = I$, and by part c, $(TT^{-1})^* = (T^{-1})^*T^*$.

Thus we have: $(T^{-1})^*T^* = I$.

That means that T^* has an inverse given by $(T^{-1})^*$.

That is, $(T^*)^{-1} = (T^{-1})^*$.