The Adjoint of a Linear Operator- HW

1. Let $F: \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation given by
$$F(a_1, a_2, a_3) = 3a_1 - 2a_2 + a_3, \quad a_1, a_2, a_3 \in \mathbb{R}.$$
Let β be the standard ordered basis for $\mathbb{R}^3$. Find a vector $v \in \mathbb{R}^3$ such that $F(a_1, a_2, a_3) =< (a_1, a_2, a_3),\ v > .$

2. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator on $\mathbb{R}^2$ given by
$$T(a_1, a_2) = (a_1 + 2a_2,\ 3a_1 - a_2)$$
with respect to the standard basis on $\mathbb{R}^2$.
   a. Find a matrix representation of T and T^*.
   b. Show by direct calculation that for all $v, w \in \mathbb{R}^2$
$$< T(v), w >=< v, T^*(w) >.$$

3. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ given by
$$T(a_1, a_2, a_3) = \big((2a_1 + a_2 - a_3),\ (a_2 + 3a_3\big),\ (3a_1 - a_2))$$
with respect to the standard basis on $\mathbb{R}^3$.
   a. Find a matrix representation of T and T^*.
   b. Show by direct calculation that for all $v, w \in \mathbb{R}^3$
$$< T(v), w >=< v, T^*(w) >.$$

4. Let T be a linear operator on a real inner product space.
   a. Let $S_1 = T + T^*$. Prove that $S_1 = S_1^*$.
   b. Let $S_2 = TT^*$. Prove that $S_2 = S_2^*$.

5. Give an example of a linear operator T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that the null space of T is NOT equal to the null space of T^*.