Inner Product Spaces- HW Problems

1. Define an inner product on $C[-\pi,\pi]$ by

$$< f, g > = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx.$$

Let $f(x) = \cos(x)$ and $g(x) = \sin(x)$.

a. Show that $f(x) = \cos(x)$ and $g(x) = \sin(x)$ are orthogonal (ie , $< f, g > = 0$).

b. Show that $\|f\| = \|g\| = 1$.

2. Define an inner product on $C[0,1]$ by

$$< f, g > = \int_{0}^{1} f(x)g(x)dx.$$

Let $f(x) = x$ and $g(x) = \sin(\pi x)$.

a. Show that $\|f + g\| \le \|f\| + \|g\|$.

b. Show that $| < f, g > | \le \|f\|\|g\|$.

3. Show that $\{\frac{1}{\sqrt{2}} < 1, 1, 0 >,\ \ \frac{1}{\sqrt{3}} < 1, -1, 1 >,\ \ \frac{1}{\sqrt{6}} < -1, 1, 2 >\}$ is an orthonormal set in R^3 with the standard inner product.

4. In a. and b., determine if each defines a norm on $C[0,1]$. If not, which properties of

i. $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$.

ii. $\|af\| = |a|\|f\|;\ a \in \mathbb{R}$

iii. $\|f + g\| \leq \|f\| + \|g\|$

does it violate.

a. $\|f\| = |f(0)| + |f(1)|$

b. $\|f\| = \max_{0 \leq x \leq 1} |f(x)|$.

Hint for problems 5, 6, 7, and 9: $\|v\| = \sqrt{< v, v >}$

5. Let V be an inner product space with $v_1, v_2 \in V$ and v_1, v_2 orthogonal. Prove that $\|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$.

6. Prove that if V is an inner product space with $v_1, v_2 \in V$ then

$$\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2.$$

7. Let $\{v_1, v_2, v_3, v_4\}$ be an orthogonal set in an inner product space V and $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Prove that

$$\left\|\sum_{i=1}^{4} a_i v_i\right\|^2 = \sum_{i=1}^{4} |a_i|^2 \|v_i\|^2.$$

8. (Consequences of the triangle inequality) $u, v \in V$, an inner product space. The triangle inequality says $\|u+v\| \le \|u\| + \|v\|$. Show

a. $\|u-v\| \le \|u\| + \|v\|$. Note: $\|-v\| = |-1|\|v\|$=$\|v\|$

b. $\|u\| \le \|u-v\| + \|v\|$ and $\|v\| \le \|u-v\| + \|u\|$.

Hint: write $u = (u-v) + v$ and $v = (v-u) + u$.

c. using part b show $|\|u\| - \|v\|| \le \|u-v\|$.

9. Let V be an inner product space. Prove that

$$< u, v > = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$$