Determinants: An Overview

For each $n \times n$ matrix, A, we can calculate a number called the determinant of A, $det(A)$. This is often written as $|A|$.

Case 1. 1×1 matrices. If $A = [a]$ then $det(A) = |A| = a$.

Case 2. 2×2 matrices. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the we have:

$$det(A) = |A| = a_{11}a_{22} - a_{21}a_{12} \,.$$

Ex. $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3(5) - (4)(-2) = 15 + 8 = 23.$

Def. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained from deleting the row and column containing a_{ij}. The determinant of (M_{ij}) is called the **Minor** of a_{ij}. We define the **Cofactor**, A_{ij}, of a_{ij} by:

$$A_{ij} = (-1)^{i+j} \det(M_{ij}).$$

Def. The **Determinant of an $n \times n$ matrix A**, where $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, is given

by: $\quad Det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$

where $A_{1j} = (-1)^{1+j} \det(M_{1j}); \quad j = 1, 2, \cdots, n.$

Case 3. 3x3 matrices. Let A be:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$;

where: $\quad A_{11} = (-1)^{1+1} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

$A_{12} = (-1)^{1+2} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} = -\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$

$A_{13} = (-1)^{1+3} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$

Ex. Find $det(A)$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 2[(1)(3) - 2(2)] - 1[4(3) - 1(2)] + 3[4(2) - 1(1)]$$

$$= 2(3 - 4) - 1(12 - 2) + 3(8 - 1)$$

$$= 2(-1) - 10 + 3(7)$$

$$= 9.$$

Ex. Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & -2 \\ -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 3 \end{bmatrix}$.

$$|A| = 1 \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ 3 & 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 0 & 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & 0 & -2 \\ -1 & 1 & 0 \\ 0 & 3 & 3 \end{vmatrix} - 3 \begin{vmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix}$$

$$= 1 \left[0 \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \right]$$

$$-3 \left[2 \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} - 0 \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} \right]$$

$$= 1[0 - 1(3 - 0) - 2(1 - 0)] - 3[2(1 - 0) - 0 + 1(-3 - 0)]$$

$$= 1[-3 - 2] - 3[2 - 3] = -5 + 3 = -2.$$

Theorem: If A is an $n \times n$ matrix with $n \geq 2$ then $det(A)$ can be expressed as a cofactor expansion using any column or row.

$Det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ (using the ith row);   or

$Det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$ (using the jth column).

Ex Find $det(A)$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

a. Using the 3rd row
b. Using the 2nd column

a. $det(A) = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$

$= 1(2-3) - 2(4-12) + 3(2-4)$

$= 1(-1) - 2(-8) + 3(-2) = -1 + 16 - 6 = 9$

b. $det(A) = (-1)^{1+2} \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}$

$= -1(12-2) + 1(6-3) - 2(4-12)$

$= -1(10) + 3 - 2(-8) = -10 + 3 + 16 = 9.$

Ex. Find $det(A)$ where $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & -4 \\ 2 & 0 & 0 & 1 \\ -2 & 3 & 0 & 1 \end{bmatrix}$ using the 2nd column.

$detA = (-1)^{1+2}(0) \begin{vmatrix} 0 & 1 & -4 \\ 2 & 0 & 1 \\ -2 & 0 & 1 \end{vmatrix} + (-1)^{2+2}(0) \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -2 & 0 & 1 \end{vmatrix}$

$+ (-1)^{3+2}(0) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ -2 & 0 & 1 \end{vmatrix} + (-1)^{4+2}(3) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 2 & 0 & 1 \end{vmatrix}$

$= 3 \left[1(1-0) - 2(-0(-8)) + 3(0-2) \right]$

$= 3\left[1 - 2(8) + 3(-2) \right] = 2(1 - 16 - 6) = 3(-21) = -63.$

Theorem If A is an $n \times n$ matrix then $det(A^t) = det(A)$

Ex. $det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = det \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

Theorem: If A is an $n \times n$ matrix which is either upper triangular or lower triangular then $det(A) =$ product of the diagonal elements.

Proof:

$$det \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$ taking cofactors along the first column

$$= a_{11} \begin{vmatrix} a_{22} & \cdots & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & \cdots & \cdots & a_{3n} \\ 0 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}a_{22\cdots}a_{nn}.$$

Ex. $det \begin{bmatrix} 1 & 5 & 7 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} = 1(2)(3) = 6$

Ex. $det \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 2 & 5 & 4 \end{bmatrix} = 2(-1)(4) = -8$.

Theorem Let A be an $n \times n$ matrix

   i.   If A has an entire row or column of zeros then $det(A) = 0$

  ii.   If A has 2 identical rows or columns then $det(A) = 0$.

Ex. $det \begin{bmatrix} 1 & 4 & 2 & 0 \\ 9 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & -3 & 2 \end{bmatrix} = 0$

$det \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 4 & 1 & 2 \\ 3 & 2 & 3 & -3 \\ 2 & 9 & 2 & 4 \end{bmatrix} = 0.$

Theorem: If A and B are $n \times n$ matrices then $\det(AB) = (\det(A))(\det(B))$.

Theorem An $n \times n$ matrix A is invertible (nonsingular) if and only if $det(A) \neq 0$. If A is invertible then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Ex. Determine if A is invertible where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

$det(A) = 1\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} + 1\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

$\qquad = 1(1-2) - 2(-2-3) + 1(-4-3)$

$\qquad = -1 - 2(-5) + (-7) = -1 + 10 - 7 = 2 \neq 0.$

So A is invertible.

Notice that an $n \times n$ matrix A is invertible precisely when the $Rank(A) = n$. Thus the n column vectors of A are linearly independent if and only if A is invertible. Hence the n column vectors of A are linearly independent if and only if $det(A) \neq 0$. So given any n vectors, $v_1, \dots, v_n$ in $\mathbb{R}^n$ we can test to see if they are linearly independent by calculating the determinant of the matrix formed by making $v_1, \dots, v_n$ the columns of matrix. If the determinant is nonzero the vectors are linearly independent. If the determinant is zero the vectors are dependent.

Ex. Determine if the vectors $v_1 = < 1, 0, -2 >$, $v_2 = < 0, 3, 1 >$, and $v_3 = < 2 - 1, 3 >$ are linearly independent.

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ -2 & 1 & 3 \end{bmatrix}$ where the columns of A are v_1, v_2, v_3.

$$det(A) = (1)\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} - (0)\begin{vmatrix} 0 & -1 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= (1)(9 + 1) + 0 + 2(0 + 6)$$

$$= 22 \neq 0.$$

Thus v_1, v_2 and v_3 are linearly independent.

Theorem: If B is a matrix obtained by interchanging any two rows or columns of an $n \times n$ matrix A then $\det(B) = -\det(A)$. In particular if two rows or columns of A are identical then $\det(A) = 0$.

Ex. $det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = -det \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

Theorem: If B is a matrix obtained by multiplying each entry of a row or column of an $n \times n$ matrix A by $k \in \mathbb{R}$, then $\det(B) = k \det(A)$. In particular, if $B = kA$ then $\det(B) = k^n \det(A)$.

Ex. $det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 21 & 24 & 27 \end{bmatrix} = 3det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}; \quad det \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} = 3^2 det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$

Theorem: If B is a matrix obtained from an $n \times n$ matrix A by adding a multiple of row i to row j or adding a multiple of column i to column j, $i \neq j$, Then $\det(B) = \det(A)$.

Thus one can use elementary row operations to put a matrix in upper (or lower) triangular form to calculate the determinant of an $n \times n$ matrix.

Ex. Evaluate $det \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix}$ using elementary row operations.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 + 4R_1 \to R_3} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 3 & -2 & 10 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 - 3R_1 \to R_4} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 4 & -4 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_3 \to R_4} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 6 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 6 & -2 \end{bmatrix}$$

$$\xrightarrow{R_4 - 2R_3 \to R_4} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

Notice that every elementary row operation we used was of type 3, thus the value of the determinant didn't change through this process. Hence:

$$det \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix} = det \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$= (1)(1)(3)(6) = 18.$$