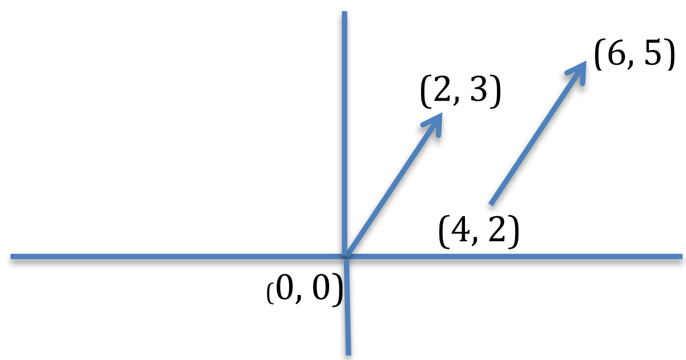
Vector Spaces

Vectors in $\mathbb{R}^2$

A nonzero vector in $\mathbb{R}^2$ can be represented by a directed line segment. So a vector is something with a magnitude, how long the vector is, and a direction.

Ex. We can think of the vector $v = < 2, 3 >$ as a line segment starting at (0, 0) (or any other point in the plane) and ending 2 units to the right and 3 units up.



The length of any vector $v = < a, b >$ in $\mathbb{R}^2$ is $|v| = \sqrt{a^2 + b^2}$

Ex. The length of $v = < 2, 3 >$ is:

$$|v| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

We can multiply any vector in $\mathbb{R}^2$ by a real number α, called a scalar, by

$$v = < a, b >$$
$$\alpha v = \alpha < a, b > = < \alpha a, \alpha b >$$

Ex. If

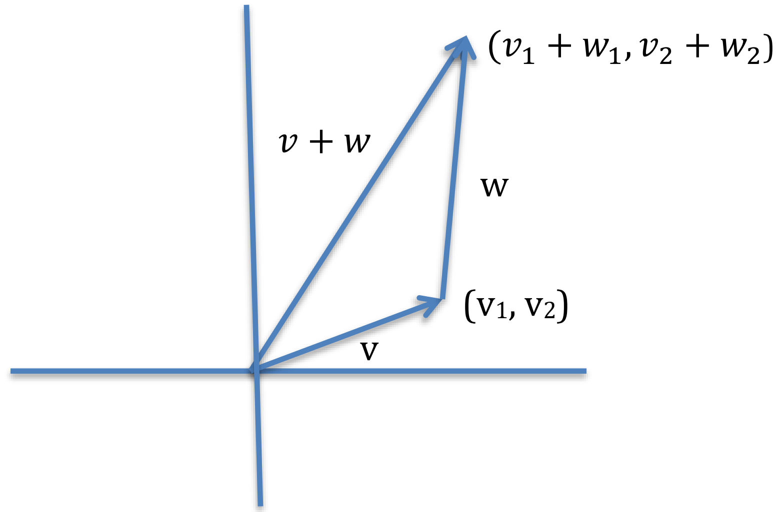
$$v = < -3,\ 2 >$$
$$3v = 3 < -3, 2 > = < -9, 6 >$$
$$-2v = -2 < -3, 2 > = < 6, -4 >$$

If we have 2 vectors:

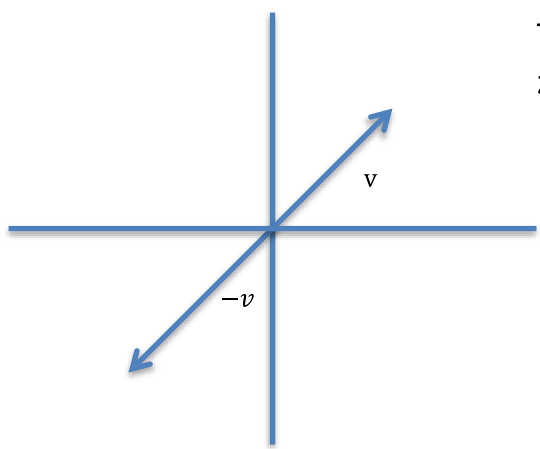
$$v = < v_1, v_2 >$$
$$w = < w_1, w_2 >$$

then $v + w = < v_1, v_2 > + < w_1, w_2 > = < v_1 + w_1, v_2 + w_2 >$.

Geometrically, $v + w$ is the vector starting at $(0,0)$ and ending at $(v_1 + w_1, v_2 + w_2)$.



If $v = < a, b >$ then $-v = < -a, -b >$.
$-v$ is the same length as v but points in the opposite direction.



Thus $v + (-v) =< 0, 0 >$, the zero vector

If w is any vector in $\mathbb{R}^2$ then $w+ < 0, 0 > = w$.

Vector Space Axioms

Def. Let V be a set (like all vectors in $\mathbb{R}^2$) on which the operations of addition and scalar multiplication (i.e. multiplying by a real number) are defined. By this we mean if $v, w \in V$ then $v + w \in V$ and $\alpha v \in V$ where α is any real number. The set V together with the operations of addition and scalar multiplication, is said to form a **Vector Space** if the following axioms hold:

A1. $v + w = w + v$ for all $v, w \in V$
A2. $(v + w) + u = v + (w + u)$ for all $u, v, w \in V$
A3. There exists an element 0 in V such that $v + 0 = v$ for every $v \in V$, (0 is the zero element)
A4. For each $v \in V$ there exists an element $-v \in V$ such that
$$v + (-v) = 0$$
A5 $1 \cdot v = v$ for all $v \in V$
A6. $(\alpha\beta)v = \alpha(\beta v)$ for any scalars $\alpha, \beta \in \mathbb{R}$ and any $v \in V$
A7. $\alpha(v + w) = \alpha v + \alpha w$ for each scalar $\alpha \in \mathbb{R}$ and any $v, w \in V$
A8. $(\alpha + \beta)v = \alpha v + \beta v$ for any scalars of $\alpha, \beta \in \mathbb{R}$ and any $v \in V$.

The elements of $\mathbb{R}$ are called **scalars**.
The elements of V are called **vectors**.

Ex. $\mathbb{R}^2$ is a vector space with (the standard)
$< v_1, v_2 > + < w_1, w_2 > = < v_1 + w_1, v_2 + w_2 >$ and
$$\alpha < v_1, v_2 > = < \alpha v_1, \alpha v_2 >.$$

To prove this we need to show $\mathbb{R}^2$ is closed under addition and scalar multiplication and verify the 8 axioms.

Let $v = < v_1, v_2 >$, $w = < w_1, w_2 >$, $u = < u_1, u_2 >$ be any vectors in $\mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$.
$\mathbb{R}^2$ is closed under addition because if $v, w \in \mathbb{R}^2$, then:
$< v_1, v_2 > + < w_1, w_2 > = < v_1 + w_1,\ v_2 + w_2 > \in \mathbb{R}^2$.

$\mathbb{R}^2$ is closed under scalar multiplication because if $v \in \mathbb{R}^2$, then:
$\alpha < v_1, v_2 > = < \alpha v_1,\ \alpha v_2 > \in \mathbb{R}^2$ for any $\alpha \in \mathbb{R}$.

A1. $< v_1, v_2 > + < w_1, w_2 > =< v_1 + w_{1,} v_2 + w_2 >$
$=< w_1 + v_1, w_2 + v_2 >$
$= < w_1, w_2 > +< v_1, v_2 >$

A2. $(< v_1, v_2 > +< w_1, w_2 >)+< u_1, u_2 >$
$=< v_1 + w_1, v_2 + w_2 > +< u_1, u_2 >$
$= < v_1 + w_1 + u_1, v_2 + w_2 + u_2 >$
$=< v_1, v_2 > +< w_1 + u_1, w_2 + u_2 >$
$= < v_1, v_2 > +(< w_1, w_2 > +< u_1, u_2 >)$

A3. $\vec{0} = < 0, 0 >$; $< v_1, v_2 > +< 0, 0 > = < v_1, v_2 >$

A4. $-< v_1, v_2 > = < -v_1, - v_2 >$
so $< v_1, v_2 > +< -v_1, -v_2 > = < 0, 0 >$

A5. $1 \cdot < v_1, v_2 > = < 1v_1, 1v_2 > =< v_1, v_2 >$

A6. $(\alpha\beta) < v_1, v_2 > = < \alpha\beta v_1, \alpha\beta v_2 > = \alpha(< \beta v_1, \beta\, v_2 >)$
$= \alpha(\beta < v_1, v_2 >)$

A7. $\alpha(< v_1, v_2 > +< w_1, w_2 >) = \alpha < v_1 + w_1, v_2 + w_2 >$
$= < \alpha(v_1 + w_1), \alpha(v_2 + w_2) >$
$=< \alpha v_1 + \alpha w_1, \alpha v_2 + \alpha w_2 >$
$=< \alpha v_1, \alpha v_2 > +< \alpha w_1, \alpha w_2 >$
$= \alpha < v_1, v_2 > +\alpha < w_1, w_2 >$

A8. $(\alpha + \beta) < v_1, v_2 > = < (\alpha + \beta)v_1, (\alpha + \beta)v_2 >$
$= < \alpha v_1 + \beta v_1, \alpha v_2 + \beta v_2 >$
$= < \alpha v_1, \alpha v_2 > + < \beta v_1, \beta v_2 >$
$= \alpha < v_1, v_2 > +\beta < v_1, v_2 >.$

So $\mathbb{R}^2$ is a vector space with this addition and scalar multiplication.

Ex. $V = \{< a, b > \in \mathbb{R}^2 | a \geq 0, b \geq 0\}$ is NOT a vector space with the standard addition and scalar multiplication.

To prove something is not a vector space we just need to show that either the set in question is not closed under addition or scalar multiplication, or one of the 8 axioms doesn't hold.
The first thing to check is whether
$v + w \in V$ whenever $v, w \in V$, and $\alpha v \in V$ for all $v \in V$ and $\alpha \in \mathbb{R}$.

In this case, $v + w \in V$ whenever $v, w \in V$, since:
$< a, b > + < c, d > = < a + c, b + d >$, and if $a, b, c, d \geq 0$ so are $a + c$ *and* $b + d$.

However, $\alpha v \notin V$ for all $v \in V$ and all $\alpha \in \mathbb{R}$. For example, if $\alpha = -1$ and $v = < 1,2 > \in V$ then $\alpha v = < -1, -2 > \notin V$.

Ex. $\mathbb{R}^n$ is a vector space with $v = < v_1, v_2 \ldots, v_n >$, $w = < w_1, w_2 \ldots, w_n >$ and $v + w = < v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n >$ and

$$\alpha v = \alpha < v_1, v_2 \ldots, v_n > = < \alpha v_1, \alpha v_2 \ldots, \alpha v_n >.$$

The proof is exactly the same as the proof for $\mathbb{R}^2$ (we just have n components to our vectors instead of 2).

A real $m \times n$ matrix (m rows, n colums) is an array of the form

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

where $a_{ij} \in \mathbb{R}$ for $i = 1, \ldots, n$; $j = 1, \ldots, m$.

Ex. $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \\ -3 & 2 & 1 \\ 4 & 3 & 5 \end{bmatrix}$ is a 4×3 matrix. The third row is $-3,2,1$ and the second column is $-1,5,2,3$.

Ex. The usual addition and scalar multiplication for matrices works as follows:

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 5 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 8 \\ 3 & 2 & 3 \end{bmatrix}$$

$$4\begin{bmatrix} -1 & -2 & 5 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -8 & 20 \\ 12 & -4 & 4 \end{bmatrix}.$$

If $m = n$ we say that A is a square matrix.

Ex. Show the set $V = M_{m\times n}(\mathbb{R})$ = all $m \times n$ matrices with real entries with the usual matrix addition and scalar multiplicaiton is a vector space.

First we show that V is closed under addition and scalar multiplication.
If $A, B \in V$ then $A + B$ is also an $m \times n$ matrix with real entries, so $A + B \in V$.

If $A \in V$ then , $\alpha \in \mathbb{R}$, is also an $m \times n$ matrix with real entries, so $\alpha A \in V$.

A1. $A + B = B + A$ for all $A, B \in V$ (matrix addition is commutative)

A2. $(A + B) + C = A + (B + C)$ for all $A, B, C \in V$ (matrix addition is associative)

A3. $0 =$ the zero matrix (zeros in all entries), so $A + 0 = A$ for all $A \in V$

A4. For each $A \in V$, $-A = (-1)A$ has the property that $A + (-A) = 0$

A5. $1 \cdot A = A$ for all $A \in V$ (property of scalar multiplication of matrices).

A6. $(\alpha\beta)A = \alpha(\beta A)$ for all $A \in V, \alpha, \beta \in \mathbb{R}$ (property of scalar multiplication of matrices)

A7. $\alpha(A + B) = \alpha A + \alpha B$ for all $A, B \in V$ and $\alpha \in \mathbb{R}$ (distributive property of scalar multiplication of matrices)

A8. $(\alpha + \beta)A = \alpha A + \beta A$ for all $A \in V$ and $\alpha, \beta \in \mathbb{R}$ (another distributive property of scalar multiplication of matrices).

So $M_{m\times n}$ is a vector space.

Ex. Let $V = P_2(\mathbb{R}) = \{all\ polynomials\ of\ degree \leq 2,\ real\ coefficients\}$.
V is a vector space with $p(x) = a_0 + a_1x + a_2x^2$
and $q(x) = b_0 + b_1x + b_2x^2$ any element of V,

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

and
$$\alpha p(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2.$$

$P_2(\mathbb{R})$ is closed under addition because:
$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \in P_2(\mathbb{R})$

$P_2(\mathbb{R})$ is closed under scalar multiplication because:
$\alpha p(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 \in P_2(\mathbb{R})$ for any $\alpha \in \mathbb{R}$.

A1. $p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$
$= b_0 + b_1x + b_2x^2 + a_0 + a_1x + a_2x^2 = q(x) + p(x)$

A2. $\big(p(x) + q(x)\big) + r(x)$
$= (a_0 + a_1x + a_2x^2 + b_0 + b_1x + b_2x^2) + c_0 + c_1x + c_2x^2$
$= (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2 + c_0 + c_1x + c_2x^2)$
$= p(x) + (\, q(x) + r(x))$

A3. $O =$ the zero polynomial i.e. $a_0,\ a_1,\ a_2$ are all 0
$q(x) + 0 = b_0 + b_1x + b_2x^2 + 0 = b_0 + b_1x + b_2x^2 = q(x)$

A4. $-p(x) = -a_0 - a_1x - a_2x^2$ so:
$p(x) + \big(-p(x)\big) = (a_0 + a_1x + a_2x^2) + (-a_0 - a_1x - a_2x^2) = 0$

A5. $1 \cdot p(x) = 1(a_0 + a_1x + a_2x^2) = a_0 + a_1x + a_2x^2 = p(x)$

A6. $(\alpha\beta)p(x) = \alpha\beta(a_0 + a_1x + a_2x^2) = \alpha(\beta a_0 + \beta a_1 x + \beta a_2 x^2)$
$= \alpha(\beta p(x))$

A7. $\alpha(p(x) + q(x)) = \alpha\ ((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2)$
$= (\alpha a_0 + \alpha b_0) + (\alpha a_1 + \alpha b_1)x + (\alpha a_2 + \alpha b_2)x^2$
$= (\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2) + (\alpha b_0 + \alpha b_1 x + \alpha b_2 x^2)$
$= \alpha p(x) + \alpha q(x)$

A8. $(\alpha+\beta)p(x) = (\alpha+\beta)(a_0 + a_1x + a_2x^2)$
$= (\alpha a_0 + \beta a_0) + (\alpha a_1 + \beta a_1)x + (\alpha a_2 + \beta a_2)x^2 = \alpha p(x) + \beta p(x).$

So V is a vector space.

In fact, $P_n(\mathbb{R})$, polynomials with real coefficients of degree $\leq n$, n a positive integer, forms a vector space.

Ex. Let $V = \{polynomials\ with\ real\ coefficients \mid\ f(0) = 0\}$.
Show that V is a vector space with the usual addition and scalar multiplication (as in the previous example).

First show that V is closed under addition.
If $f(x), g(x) \in V$ then $f(0) = g(0) = 0$.
Then $h(x) = f(x) + g(x)$ has $h(0) = f(0) + g(0) = 0$.
Since the sum of two polynomials is also a polynomial, $h(x) \in V$.

Now show that V is closed under scalar multiplication.
If $f(x) \in V$ and $c \in \mathbb{R}$ then let $h(x) = cf(x)$.
$h(0) = cf(0) = 0$ and the product of a real number and a polynomial is again a polynomial. Thus $h(x) \in V$.

So V is closed under addition and scalar multiplication.

$f(x) = 0 \in V$ is the additive identity and since $f(0) = 0$ and $f(x)$ is a polynomial with real coefficients.

If $f(x) \in V$, then the additive inverse, $-f(x) \in V$, since $-f(0) = 0$ and -1 times a polynomial is again a polynomial and $f(x) + \left(-f(x)\right) = 0$.

V satisfies axioms $1 - 8$ as in the previous example, so V is a vector space.

Let $\mathfrak{J} = \{functions\ from\ \mathbb{R}\ to\ \mathbb{R}\}$. So the "vectors" in $\mathfrak{J}$ are functions from $\mathbb{R}$ to $\mathbb{R}$ (e.g., $f(x) = x^2$, $g(x) = cosx$, etc.).

Vector addition is just the usual addition of functions. For example, $f(x) = x^2 - 3x$, $g(x) = 2x^2 + 1$ are in $\mathfrak{J}$. $f(x) + g(x) = 3x^2 - 3x + 1$.

Scalar multiplication is defined as the usual multiplication of a constant times a function. For example, $f(x) = x^2 - 3x \in \mathfrak{J}$, $4 \in \mathbb{R}$, $4f(x)) = 4x^2 - 12x$.

Ex. Show that $\mathfrak{J} = \{functions\ from\ \mathbb{R}\ to\ \mathbb{R}\}$ with the usual addition and scalar multiplication is a vector space.

$\mathfrak{J}$ is closed under addition since if $f(x), g(x) \in \mathfrak{J}$ then $f(x) + g(x) \in \mathfrak{J}$ because the sum of two functions from $\mathbb{R}$ to $\mathbb{R}$ is again a function from $\mathbb{R}$ to $\mathbb{R}$.

$\mathfrak{J}$ is closed under scalar multiplication because a constant multiple of a function from $\mathbb{R}$ to $\mathbb{R}$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

The zero vector in $\mathfrak{J}$ is the function $f(x) = 0$.

If $f(x) \in \mathfrak{J}$ then its additive inverse $-f(x) \in \mathfrak{J}$.

Since axioms 1-8 are satisfied by real numbers they are also satisfied by $\mathfrak{J}$ with the usual addition and scalar multiplication of functions.

Thus $\mathfrak{J}$ is a vector space.

Ex. Let $V = \{polynomials\ with\ real\ coefficients |\ f(0) = 1\}$ with the usual addition and scalar multiplication for functions. Show that V is not a vector space.

Notice that V is not closed under addition or scalar multiplication since if $f(x), g(x) \in V$ then $h(x) = f(x) + g(x) \notin V$ since
$h(0) = f(0) + g(0) = 1 + 1 = 2.$

$h(x) = 3(f(x))$ then $h(0) = 3\big(f(0)\big) = 3.$

In addition, there is no additive identity (i.e. a zero vector) since if $g(x)$ is the 0 vector then $f(x) + g(x) = f(x)$. But then $g(0) = 0$. Thus $g(x) \notin V$.

There is no additive inverse as well. If $f(x) \in V$ and $g(x)$ is the additive inverse of $f(x)$, then $f(x) + g(x) = 0$. But then $g(x) = -f(x)$ and $g(0) = -f(0) = -1$. Thus $g(x) \notin V$.

Ex. Let $V = \mathbb{R}^2$ and define vector addition by

$$< a_1, a_2 > + < b_1, b_2 > = < a_1 - b_1,\ a_2 + b_2 >$$

and scalar multiplication by $c < a_1, a_2 > = < ca_1,\ ca_2 >$. Show that V is not a vector space.

It's straightforward to see that V is closed under addition and scalar multiplication. However, several of the axioms of vector spaces don't hold.

Axiom 1: $v + w = w + v$.
If we let $v = < a_1, a_2 >$, $w = < b_1, b_2 >$, then

$$v + w = < a_1 - b_1,\ a_2 + b_2 >$$
$$w + v = < b_1 - a_1,\ a_2 + b_2 >$$

and $a_1 - b_1 \neq b_1 - a_1$ for all $a_1, b_1 \in \mathbb{R}^2$. So $v + w \neq w + v$.

Axiom 2: $(v + w) + z = v + (w + z)$.
If we let $v =< a_1, a_2 >$, $w =< b_1, b_2 >$, $z =< d_1, d_2 >$ then

$$(v + w) + z =< a_1 - b_1,\ a_2 + b_2 > + < d_1, d_2 >$$
$$=< a_1 - b_1 - d_1,\ a_2 + b_2 + d_2 >$$

$$v + (w + z) =< a_1, a_2 > + < b_1 - d_1,\ b_2 + d_2 >$$
$$=< a_1 - (b_1 - d_1),\ a_2 + (b_2 + d_2) >$$
$$=< a_1 - b_1 + d_1, a_2 + b_2 + d_2 >$$

So $(v + w) + z \neq v + (w + z)$.

Axiom 8: If $a, b \in \mathbb{R}$ and $v \in V$ then $(a + b)v = av + bv$.
If we let $v =< a_1, a_2 >$ then

$$(a + b)v = (a + b) < a_1, a_2 >=< (a + b)a_1, (a + b)a_2 >$$

$$av + bv = a < a_1, a_2 > +b < a_1, a_2 >$$
$$=< aa_1, aa_2 > + < ba_1, ba_2 >$$
$$=< aa_1 - ba_1, aa_2 + ba_2 >$$
$$=< (a - b)a_1, (a + b)a_2 >$$

So $(a + b)v \neq av + bv$.

It is possible to have a nonstandard definition of vector addition and scalar multiplication on $V = \mathbb{R}^2$ for which V is a vector space. One example is:

If $v =< a_1, a_2 >$, $w =< b_1, b_2 >$ then
$v + w =< a_1 + b_1 - 1,\ a_2 + b_2 >$ and $cv =< ca_1 - c + 1, ca_2 >$.

However, notice that in this case the zero vector is $< 1,0 >$ not $< 0,0 >$ and the additive inverse of $< a_1, a_2 >$ is $< 2 - a_1, -a_2 >$ not $< -a_1, -a_2 >$.

Theorem (cancellation law for vector addition): If v, w, and z are vectors in a vector space V and $v + z = w + z$ then $v = w$.

Proof: There exists a vector $u \in V$ such that $z + u = 0$. Thus

$$\begin{aligned} v &= v + 0 \\ &= v + (z + u) \\ &= (v + z) + u \\ &= (w + z) + u \\ &= w + (z + u) \\ &= w + 0 \\ &= w. \end{aligned}$$

Corollary: The zero vector is unique.

Proof: Suppose v and w are both zero vectors. Then

$$z + v = z$$
$$z + w = z$$

Thus: $z + v = z + w$.

By the cancellation law: $v = w$.

Corollary: If $v \in V$ then its additive inverse is unique.

Proof: Suppose w_1, w_2 are both additive inverses of $v \in V$, then

$$v + w_1 = 0$$
$$v + w_2 = 0.$$

Thus: $v + w_1 = v + w_2$.

By the cancellation law: $w_1 = w_2$.

Ex. Show $V = \{< a, 3 >\in \mathbb{R}^2 | \; a \in \mathbb{R}\}$ with:

$< a, b > + < c, d > = < a + c, b + d >$ and $\alpha < a, b > = < \alpha a, \alpha b >$

is not a vector space.

First check if V is closed under addition and scalar multiplication.

$$v, w \in V, \qquad v = < v_1, 3 >, \quad w = < w_1, 3 >$$

$$v + w = < v_1 + w_1, 6 > \notin V$$

So V is not closed under addition.

Also if $\alpha = 3$, for example,

$$\alpha v = 3 < v_1, 3 > = < 3v_1, 9 > \notin V$$

So V is not closed under scalar multiplication either.

Ex. Let $V = \{(x, y) \in \mathbb{R}^2 | \; y = 3x\}$. Show that V is a vector space with the usual vector addition and scalar multiplication.

V is closed under addition since if $v, w \in V$ then for some $x_1, x_2 \in \mathbb{R}$

$v = < x_1, 3x_1 >$

$w = < x_2, 3x_2 >$ and

$v + w = < x_1, 3x_1 > + < x_2, 3x_2 > = < x_1 + x_2, 3(x_1 + x_2) > \in V$.

V is closed under scalar multiplication since if $v \in V$ and $c \in \mathbb{R}$ then

$v = < x_1, 3x_1 >$

$cv = c < x_1, 3x_1 >$

$= < cx_1, 3cx_1 > \in V$.

The zero vector in V is: $< 0,0 > = < 0,3(0) > \in V$.

V contains all additive inverses since if $v \in V$ and $v = < x_1, 3x_1 >$ then $w = < -x_1, 3(-x_1) > \in V$ is its additive inverse since:

$v + w = < x_1, 3x_1 > + < -x_1, 3(-x_1) >$

$= < 0,0 >$.

It's straightforward to check that the other axioms hold.

Ex. Show that $V = \{(x, y) \in \mathbb{R}^2 |\ y = 3x + 1\}$ is not a vector space under the usual vector addition and scalar multiplication.

V is not closed under addition since if $v, w \in V$ and $v =< x_1, 3x_1 + 1 >$ $w =< x_2, 3x_2 + 1 >$ then

$$v + w =< x_1, 3x_1 + 1 > + < x_2, 3x_2 + 1 >.$$
$$=< x_1 + x_2,\ 3(x_1 + x_2) + 2 > \notin V.$$

V is also not closed under scalar multiplication since if $c \in \mathbb{R}$, $c \neq 1$ then

$$cv =< cx_1, c(3x_1 + 1) >$$
$$=< cx_1, 3(cx_1) + c > \neq < cx_1, 3(cx_1) + 1 >$$

So $cv \notin V$.

The zero vector is not in V. If w is the zero vector then $w + v = v$ for all $v \in V$.
But by usual vector addition that means $w =< 0,0 >$.
However, $< 0,0 > \notin V$ since $< 0,0, > \neq < 0,3(0) + 1 > = < 0,1 >$.

Additive inverses are not in V.
If $v \in V$ then w is an additive inverse of v if $v + w =< 0,0 >$
Thus if $v =< x_1, 3x_1 + 1 >$,
then $w =< -x_1, 3(-x_1) - 1 > = < -x_1, -3x_1 - 1 >$
since $< x_1, 3x_1 + 1 > + < -x_1, -3x_1 - 1 > = < 0,0 >$.
But $< -x_1, -3x_1 - 1 > \notin V$.

Ex. Let $V = \{2x2\ matrices, A, where\ \det(A) = 0\}$
Let the addition and scalar multiplication be the usual matrix operations.
Show V is not a vector space.

We know V is closed under scalar multiplication because
$\det(\alpha A) = \alpha^2 \det(A)$, since A is $2x2$, and $\det(A) = 0$,
$\alpha^2 \det(A) = 0$.

However, $\det(A + B)$ is not necessarily 0, if $\det(A)$ and $\det(B) = 0$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $\quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det(A) = 0, \det(B) = 0\ but\ \det(A + B) = det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0.$$

So V is not closed under addition.