The Differential of a Map

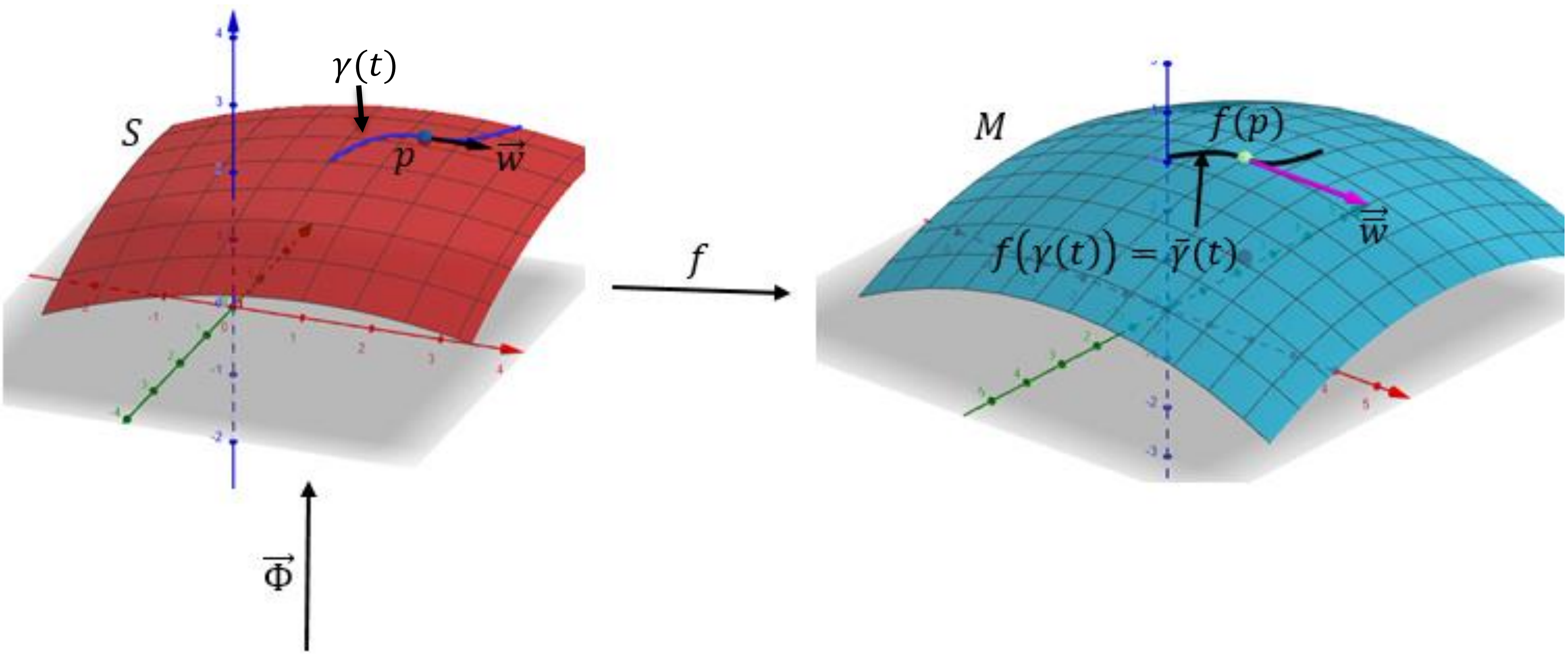
Let $f\colon S \to M$ be a differentiable map between differentiable manifolds. We define $Df_p\,;\ p \in S$, the differential of f at p, as a linear transformation between tangent spaces.

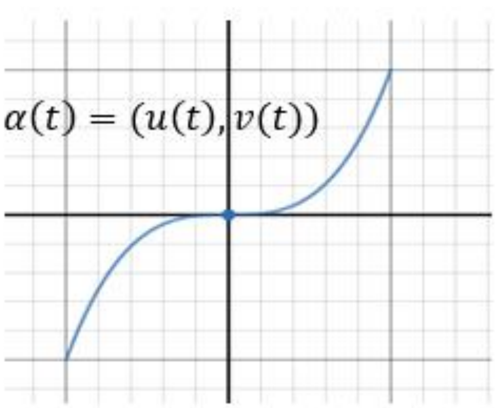
$$Df_p\colon T_pS \to T_{f(p)}M$$

Given any vector, $\vec{w} \in T_pS$, there exists a curve, γ, in S passing through p, i.e. $\gamma(t_0) = p \in S$, such that $\gamma'(t_0) = \vec{w}$. Then $\bar{\gamma}(t) = f\big(\gamma(t)\big)$ is a curve in M passing through $f(p)$ at $t = t_0$.

Let $\vec{\bar{w}} = \bar{\gamma}'(t_0) \in T_{f(p)}M$. Then we define $Df_p(\vec{w})$ by:

$$Df_p(\vec{w}) = \vec{\bar{w}} \in T_{f(p)}M$$



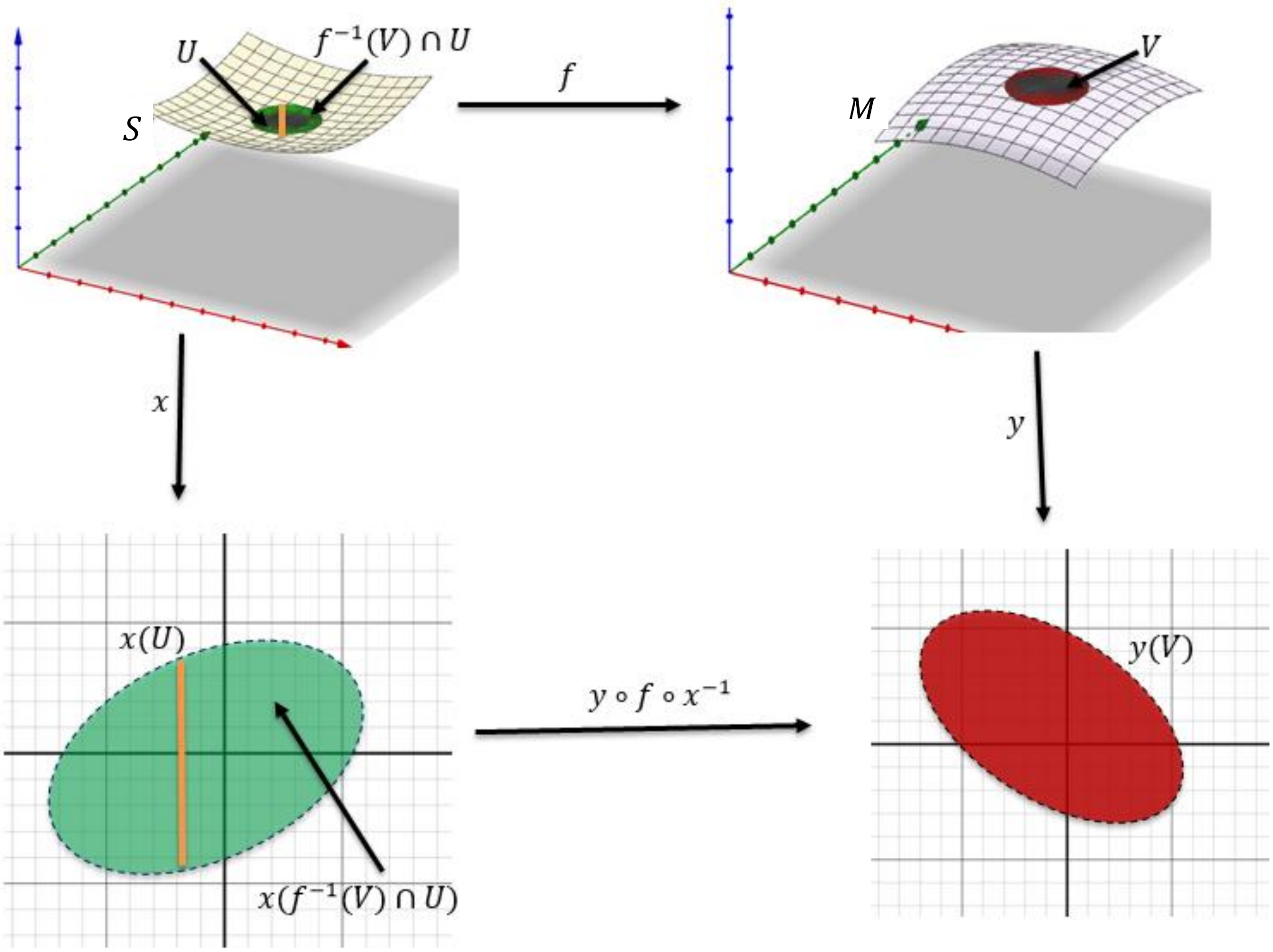


One can check that this definition does not depend on which curve γ in S one chooses such that $\gamma(t_0) = p \in S$ and $\gamma'(t_0) = \vec{w}$.

How do we calculate Df_p?

If we let $x: U \subseteq S \to \mathbb{R}^m$ be a local coordinate chart on $U \subseteq S$ and $y: V \subseteq M \to \mathbb{R}^n$ be a local coordinate chart on $V \subseteq M$, then:

$$y \circ f \circ x^{-1}: x\big(U \cap f^{-1}(V)\big) \subseteq \mathbb{R}^m \to y(V) \subseteq \mathbb{R}^n$$



So $y \circ f \circ x^{-1}$ maps an open set in $\mathbb{R}^m$ into an open set in $\mathbb{R}^n$ by

$$y \circ f \circ x^{-1}(x^1, \dots, x^m) = (y^1, \dots, y^n) = \big(y^1(x^1, \dots, x^m), \dots, y^n(x^1, \dots, x^m)\big).$$

The standard basis for T_pS is given by $\left\{\frac{\partial}{\partial x^i}\right\} = \{\frac{\partial x^{-1}}{\partial x^i}\}$ and $\left\{\frac{\partial}{\partial y^j}\right\} = \{\frac{\partial y^{-1}}{\partial y^j}\}$ on $T_{f(p)}M$, so we can write:

$$Df_p = \left(\frac{\partial y^j}{\partial x^i}\right)_p ; \quad 1 \leq i \leq m, \quad 1 \leq j \leq n$$

where this matrix is a mapping with respect to $\left\{\frac{\partial}{\partial x^i}\right\}$ and $\left\{\frac{\partial}{\partial y^j}\right\}$.

Ex. Let S be the surface in $\mathbb{R}^3$ given by:

$$x^{-1}(x^1, x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2); \quad x^1, x^2 \in \mathbb{R}.$$

Let $M = \{(y^1, y^2, y^3) \in \mathbb{R}^3 | \ (y^1)^2 + (y^2)^2 + (y^3)^2 = 1\,,\ y^3 > 0\}$
be the upper hemisphere.

$f: S \to M$ by:

$$f(x^1, x^2, (x^1)^2 + (x^2)^2) = \left(\frac{-2x^1}{\sqrt{1+4(x^1)^2+4(x^2)^2}}, \frac{-2x^2}{\sqrt{1+4(x^1)^2+4(x^2)^2}}, \frac{1}{\sqrt{1+4(x^1)^2+4(x^2)^2}}\right)$$

f is called the Gauss map.

Now let $y: M \to \mathbb{R}^2$ by $y\left(y^1, y^2, \sqrt{1-(y^1)^2-(y)^2}\right) = (y^1, y^2)$.

Find df.

$$y \circ f \circ x^{-1}(x^1, x^2) = \left(\frac{-2x^1}{\sqrt{1+4(x^1)^2+4(x^2)^2}}, \frac{-2x^2}{\sqrt{1+4(x^1)^2+4(x^2)^2}} \right)$$

$$= \left(y^1(x^1, x^2), y^2(x^1, x^2) \right).$$

$$df = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} \end{pmatrix}$$

$$\frac{\partial y^1}{\partial x^1} = \frac{-2\left(1+4\left(x^2\right)^2\right)}{(1+4(x^1)^2+4(x^2)^2)^{\frac{3}{2}}} \qquad \frac{\partial y^1}{\partial x^2} = \frac{\partial y^2}{\partial x^1} = \frac{8\left(x^1\right)\left(x^2\right)}{(1+4(x^1)^2+4(x^2)^2)^{\frac{3}{2}}}$$

$$\frac{\partial y^2}{\partial x^2} = \frac{-2\left(1+4\left(x^1\right)^2\right)}{(1+4(x^1)^2+4(x^2)^2)^{\frac{3}{2}}}.$$

$$df = \frac{-2}{(1+4(x^1)^2+4(x^2)^2)^{\frac{3}{2}}} \begin{pmatrix} 1+4(x^2)^2 & -4(x^1)(x^2) \\ -4(x^1)(x^2) & 1+4(x^1)^2 \end{pmatrix}.$$

Where the basis for $T_p(S)$ is given by:

$$\left\{ \frac{\partial x^{-1}}{\partial x^1}, \frac{\partial x^{-1}}{\partial x^2} \right\} = \{< 1, 0, 2x^1 >, < 0, 1, 2x^2 >\}$$

and the basis for $T_{f(p)}(M)$ is given by:

$$\left\{ \frac{\partial y^{-1}}{\partial y^1}, \frac{\partial y^{-1}}{\partial y^2} \right\} = \left\{ < 1, 0, \frac{-y^1}{\sqrt{1-\left(y^1\right)^2-\left(y^2\right)^2}} >, \ < 0, 1, \frac{-y^2}{\sqrt{1-\left(y^1\right)^2-\left(y^2\right)^2}} > \right\}$$

since $y^{-1}(y^1, y^2) = \left(y^1, y^2, \sqrt{1-(y^1)^2-(y^2)^2} \right)$.

As an example, let's take the point on S given by $p = (1, 2, 5)$,
i.e. $x^1 = 1, \; x^2 = 2$
and take a vector $\vec{w} \in T_{(1,2,5)}S$ given by

$$\vec{w} = 3\left(\frac{\partial x^{-1}}{\partial x^1}\right) + \left(\frac{\partial x^{-1}}{\partial x^2}\right) \qquad \text{at } (1, 2, 5).$$

where $\quad \dfrac{\partial x^{-1}}{\partial x^1} =< 1, 0, 2x^1 >, \qquad \dfrac{\partial x^{-1}}{\partial x^2} =< 0, 1, 2x^2 >$

are the basis vectors for T_pS.

So at $x^1 = 1, \; x^2 = 2$ we have:

$$\frac{\partial x^{-1}}{\partial x^1} =< 1, 0, 2 >, \qquad \frac{\partial x^{-1}}{\partial x^2} =< 0, 1, 4 >.$$

And in the standard basis for $\mathbb{R}^3$,

$$\vec{w} = 3 < 1, 0, 2 > + < 0, 1, 4 > = < 3, 1, 10 >.$$

Plugging in $\quad x^1 = 1, \quad x^2 = 2 \quad$ into Df, we get:

$$\left(Df_{(1,2,5)}\right)(\vec{w}) = \frac{-2}{(21)^{\frac{3}{2}}}\begin{pmatrix} 17 & -8 \\ -8 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{-2}{(21)^{\frac{3}{2}}}\begin{pmatrix} 43 \\ -19 \end{pmatrix}.$$

Thus $\vec{w}$ gets mapped to the vector in

$$T_{f(1,2,5)}M = T_{\left(\frac{-2}{\sqrt{21}},\frac{-4}{\sqrt{21}},\frac{1}{\sqrt{21}}\right)}M,$$

given by $\frac{-2}{(21)^{\frac{3}{2}}} < 43, -19 >$ with respect to the basis:

$$\left\{\frac{\partial y^{-1}}{\partial y^1}, \frac{\partial y^{-1}}{\partial y^2}\right\} = \left\{< 1, 0, \frac{-y^1}{\sqrt{1-(y^1)^2-(y^2)^2}} >,\ < 0, 1, \frac{-y^2}{\sqrt{1-(y^1)^2-(y^2)^2}} >\right\}.$$

$$f(p) = \left(\frac{-2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}, \frac{1}{\sqrt{21}}\right)$$

so in this case $y^1 = \frac{-2}{\sqrt{21}}$ and $y^2 = \frac{-4}{\sqrt{21}}$.

Thus, $\frac{\partial y^{-1}}{\partial y^1} =< 1, 0, 2 >$ and $\frac{\partial y^{-1}}{\partial y^2} =< 0, 1, 4 >$ so we can write:

$$Df_{(1,2,5)}(\vec{w}) = \frac{-2}{(21)^{\frac{3}{2}}} < 43, -19 >= \frac{-2}{(21)^{\frac{3}{2}}}\left(43\frac{\partial y^{-1}}{\partial y^1} - 19\frac{\partial y^{-1}}{\partial y^2}\right)$$

$$= \frac{-2}{(21)^{\frac{3}{2}}}(43 < 1, 0, 2 > -19 < 0, 1, 4 >)$$

$$= \frac{-2}{(21)^{\frac{3}{2}}} < 43, -19, 10 >$$

in the standard basis for $\mathbb{R}^3$.