Riemannian Metrics- Length and Volume- HW Problems

1. Find the length of the portion of the great circle on the unit sphere in $\mathbb{R}^3$ starting at $(0,0,-1)$ and ending at $(1,0,0)$ using the metric:

a. Induced by the inverse of the stereographic projection

$$\vec{\Phi}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

First show this metric is $(h_{ij}) = \frac{4}{(u^2+v^2+1)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

b. If (h_{ij}) are the components of the metric in part a then let the components of the metric be given by $g_{ij} = \frac{1}{(u^2+v^2+1)^2} h_{ij}$.

2. Find the surface area of the lower hemisphere of the unit sphere $\{(x,y,z) \in \mathbb{R}^3 |\, x^2 + y^2 + z^2 = 1,\ z < 0\}$, using

   a. the metric in problem 1a

   b. the metric in problem 1b.

3. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, with the metric given by

$$\left(g_{ij}\right) = \frac{4}{\left(1-x^2-y^2\right)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

   a. Find the length of the curve $x^2 + y^2 = \frac{1}{4}$.

   b. Find the area of the region bounded by $x^2 + y^2 = \frac{1}{4}$, i.e., $x^2 + y^2 < \frac{1}{4}$.