Normal Curvature and Geodesic Curvature

The shape of a surface will clearly impact the curvature of the curves on the surface. For example, it's possible for a curve in a plane or on a cylinder to have zero curvature everywhere (i.e. it's a line or a portion of a line). However, it's not possible for a curve on a sphere to have zero curvature everywhere. So one way to measure how much a surface curves is by examining the curvature of the curves on the surface, this will lead us to the second fundamental form.

Let γ be a unit speed curve on an oriented surface, S. Then, $\gamma'(s)$ is a unit vector that is tangent to the surface. Thus, $\gamma'(s)$ is perpendicular to the unit normal vector, $\vec{N}$, of S. So $\gamma'(s)$, $\vec{N}$, and $\vec{N} \times \gamma'(s)$ are mutually perpendicular unit vectors.

Since $\gamma' \cdot \gamma' = 1$, by differentiating this equation we get:

$$\gamma''(s) \cdot \gamma'(s) = 0.$$

Thus, $\gamma''(s)$ is perpendicular to $\gamma'(s)$ and must lie in the plane spanned by $\vec{N}$ and $\vec{N} \times \gamma'(s)$. So we can write:

$$\gamma''(s) = a\vec{N} + b\left(\vec{N} \times \gamma'(s)\right).$$

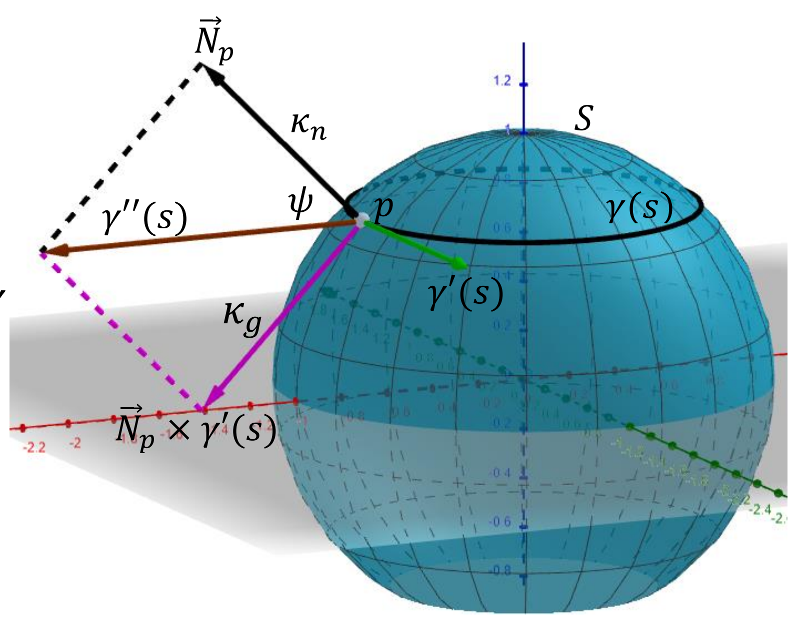
Def. We define

$a = \kappa_n =$ the **normal curvature** of γ

$b = \kappa_g =$ the **geodesic curvature** of γ

so:

$$\gamma''(s) = \kappa_n\vec{N} + \kappa_g\left(\vec{N} \times \gamma'(s)\right).$$



Notice that if we replace $\vec{N}$ with $-\vec{N}$ (the other unit normal of S) the normal and geodesic curvature also change signs.

Proposition: $\kappa_n = \gamma''(s) \cdot \vec{N}$

$$\kappa_g = \gamma''(s) \cdot \left(\vec{N} \times \gamma'(s)\right)$$

$\kappa^2 = \kappa_n^2 + \kappa_g^2$; where $\kappa =$ curvature of γ

and

$$\kappa_n = \kappa \cos\psi, \quad \kappa_g = \pm\kappa \sin\psi$$

where ψ is the angle between $\vec{N}$ and the principal normal $\vec{n}$.

Recall that the principal normal, $\vec{n}$, is defined by $\vec{n} = \frac{1}{\kappa(s)}\gamma''(s)$.

Proof:

$$\gamma''(s) = \kappa_n \vec{N} + \kappa_g \left(\vec{N} \times \gamma'(s)\right)$$

$$\gamma''(s) \cdot \vec{N} = \left(\kappa_n \vec{N} + \kappa_g \left(\vec{N} \times \gamma'(s)\right)\right) \cdot \vec{N} = \kappa_n$$

$$\gamma''(s) \cdot \left(\vec{N} \times \gamma'(s)\right) = \left(\kappa_n \vec{N} + \kappa_g \left(\vec{N} \times \gamma'(s)\right)\right) \cdot \left(\vec{N} \times \gamma'(s)\right) = \kappa_g$$

$$\kappa^2 = \|\gamma''(s)\|^2 = \left(\kappa_n \vec{N} + \kappa_g \left(\vec{N} \times \gamma'(s)\right)\right) \cdot \left(\kappa_n \vec{N} + \kappa_g \left(\vec{N} \times \gamma'(s)\right)\right)$$

$$= \kappa_n^2 + \kappa_g^2 .$$

Since $\kappa(s)\vec{n} = \gamma''(s)$, we have:

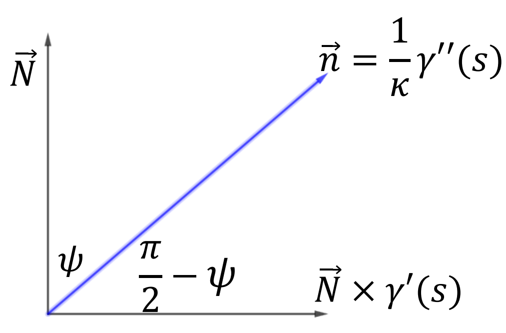
$$\kappa(s)\vec{n} = \kappa_n\vec{N} + \kappa_g\left(\vec{N} \times \sigma'(s)\right)$$

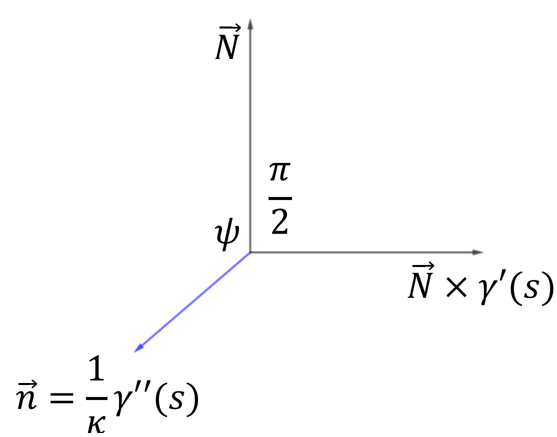
Given any two vectors, $\vec{w}_1$ and $\vec{w}_2$, $\vec{w}_1 \cdot \vec{w}_2 = \|\vec{w}_1\|\,\|\vec{w}_2\|\cos\psi$ where ψ is the angle between $\vec{w}_1$ and $\vec{w}_2$.

So since
$$\begin{aligned}\kappa_n &= \gamma''(s) \cdot \vec{N}\\ &= \left(\kappa(s)\right)\vec{n} \cdot \vec{N}\\ \kappa_n &= \kappa\cos\psi\end{aligned}$$

where ψ is the angle between the principal normal, $\vec{n}$, and $\vec{N}$.

$$\begin{aligned}\kappa_g &= \gamma''(s) \cdot \left(\vec{N} \times \gamma'(s)\right)\\ &= \left(\kappa(s)\right)\vec{n} \cdot \left(\vec{N} \times \gamma'(s)\right)\\ &= \kappa\cos\left(\frac{\pi}{2} - \psi\right) \text{ or } \kappa\cos\left(\frac{\pi}{2} + \psi\right); \quad \text{depending on } \vec{n}\end{aligned}$$





$$\kappa_g = \pm\kappa\sin\psi.$$

Proposition: If γ is a unit speed curve on an oriented surface parametrized by $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S$ and $\gamma(s) = \vec{\Phi}\big(u(s), v(s)\big)$, then

$$\kappa_n = < W(\gamma'(s)), \gamma'(s) >$$

or

$$\kappa_n = L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$$

where $L = \vec{\Phi}_{uu} \cdot \vec{N}$, $M = \vec{\Phi}_{uv} \cdot \vec{N} = \vec{\Phi}_{vu} \cdot \vec{N}$, and $N = \vec{\Phi}_{vv} \cdot \vec{N}$.

Proof: $\gamma'(s)$ is tangent to S so it's perpendicular to $\vec{N}$. Hence,

$\vec{N} \cdot \gamma'(s) = 0.$ Differentiating we get:

$$\vec{N} \cdot \gamma''(s) + \vec{N}' \cdot \gamma'(s) = 0$$

$$\kappa_n = \vec{N} \cdot \gamma''(s) = -\vec{N}' \cdot \gamma'(s).$$

But we know:

$$\vec{N}'(s) = \frac{d}{ds}\Big(\tilde{G}\big(\gamma(s)\big)\Big) = -W\big(\gamma'(s)\big).$$

So: $\kappa_n = W\big(\gamma'(s)\big) \cdot \gamma'(s).$

We saw earlier this is just: $\kappa_n = L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$

Thus, any two curves on a surface, S, that go through the point $p \in S$ and have parallel tangent vectors at $p \in S$ must have the same normal curvature at $p \in S$.

Ex. Let γ be a regular curve but not necessarily unit speed. Show that if $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S$ is a parametrization of S and $\gamma(t) = \vec{\Phi}(u(t), v(t))$, then:

$$\kappa_n = \frac{L\left(\frac{du}{dt}\right)^2 + 2M\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + N\left(\frac{dv}{dt}\right)^2}{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}$$

where $E = \vec{\Phi}_u \cdot \vec{\Phi}_u, \quad F = \vec{\Phi}_u \cdot \vec{\Phi}_v, \quad G = \vec{\Phi}_v \cdot \vec{\Phi}_v$ (i.e. the denominator is $\gamma'(t) \cdot \gamma'(t) = \left(\frac{ds}{dt}\right)^2$) and

$$\kappa_g = \frac{\gamma''(t) \cdot \left(\vec{N} \times \gamma'(t)\right)}{\left(E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2\right)^{\frac{3}{2}}} .$$

We know that if γ is unit speed, then:

$$\kappa_n = L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2 .$$

By the chain rule:

$$\frac{du}{dt} = \frac{du}{ds}\frac{ds}{dt}$$

so: $$\frac{du}{ds} = \frac{\frac{du}{dt}}{\left(\frac{ds}{dt}\right)} \text{ and } \frac{dv}{ds} = \frac{\frac{dv}{dt}}{\left(\frac{ds}{dt}\right)} .$$

Thus we have:

$$\kappa_n = L\left(\frac{\frac{du}{dt}}{\left(\frac{ds}{dt}\right)}\right)^2 + 2M\left(\frac{\frac{du}{dt}}{\left(\frac{ds}{dt}\right)}\right)\left(\frac{\frac{dv}{dt}}{\left(\frac{ds}{dt}\right)}\right) + N\left(\frac{\frac{dv}{dt}}{\left(\frac{ds}{dt}\right)}\right)^2$$

$$= \frac{1}{\left(\frac{ds}{dt}\right)^2}\left(L\left(\frac{du}{dt}\right)^2 + 2M\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + N\left(\frac{dv}{dt}\right)^2\right)$$

and

$$\left(\frac{ds}{dt}\right)^2 = \left(\gamma'(t)\cdot\gamma'(t)\right) = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2.$$

$$\Longrightarrow \qquad \kappa_n = \frac{L\left(\frac{du}{dt}\right)^2 + 2M\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + N\left(\frac{dv}{dt}\right)^2}{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}.$$

For a unit speed curve we have:

$$\kappa_g = \gamma''(s)\cdot\left(\vec{N}\times\gamma'(s)\right)$$

By the chain rule:

$$\frac{d\gamma}{ds} = \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}$$

$$\frac{d^2\gamma}{ds^2} = \frac{\frac{ds}{dt}\left(\frac{d}{ds}\left(\frac{d\gamma}{dt}\right)\right) - \left(\frac{d\gamma}{dt}\right)\left(\frac{d}{ds}\left(\frac{ds}{dt}\right)\right)}{\left(\frac{ds}{dt}\right)^2}$$

$$\frac{d^2\gamma}{ds^2} = \frac{\frac{ds}{dt}\left(\frac{d^2\gamma}{dt^2}\Big/\frac{ds}{dt}\right) - \frac{d\gamma}{dt}\left(\frac{d^2 s}{dt^2}\Big/\frac{ds}{dt}\right)}{\left(\frac{ds}{dt}\right)^2}.$$

Now by substituting into the formula for κ_g:

$$\kappa_g = \frac{d^2\gamma}{ds^2}\cdot\left(\vec{N}\times\frac{d\gamma}{ds}\right) = \left[\frac{\frac{ds}{dt}\left(\frac{d^2\gamma}{dt^2}\right) - \left(\frac{d\gamma}{dt}\right)\left(\frac{d^2 s}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^3}\right]\cdot\left(\vec{N}\times\frac{d\gamma}{dt}\Big/\frac{ds}{dt}\right).$$

Since $\frac{d\gamma}{dt}\cdot\left(\vec{N}\times\frac{d\gamma}{dt}\right) = 0$, we get:

$$\kappa_g = \frac{\frac{ds}{dt}\left(\frac{d^2\gamma}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^3}\cdot\left(\frac{1}{\frac{ds}{dt}}\right)\left(\vec{N}\times\frac{d\gamma}{dt}\right) = \frac{\frac{d^2\gamma}{dt^2}\cdot\left(\vec{N}\times\frac{d\gamma}{dt}\right)}{\left(\frac{ds}{dt}\right)^3}.$$

$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$ so we can write:

$$\kappa_g = \frac{\gamma''(t)\cdot\left(\vec{N}\times\gamma'(t)\right)}{\left(E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2\right)^{\frac{3}{2}}}.$$

Ex. Show that the normal curvature of any curve on a sphere of radius R is $\pm\frac{1}{R}$.

Using the previous example, we know:

$$\kappa_n = \frac{L\left(\frac{du}{dt}\right)^2 + 2M\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + N\left(\frac{dv}{dt}\right)^2}{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}.$$

For a sphere of radius R, using spherical coordinates, the first fundamental form is:

$$\begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \varphi \end{pmatrix}$$

(We calculated this for $R = 1$ earlier. A similar calculation gives this result.)

and the second fundamental form (as we calculated) is:

$$\begin{pmatrix} -R & 0 \\ 0 & -R \sin^2 \varphi \end{pmatrix}.$$

$$\kappa_n = \frac{-R(u')^2 - R\sin^2\varphi (v')^2}{R^2(u')^2 + R^2 \sin^2 \varphi (v')^2} = -\frac{1}{R}.$$

If we switch the orientation of the sphere by taking $-\vec{N}$ instead of $\vec{N}$, the first fundamental form is unchanged but the second fundamental form is multiplied by -1.

So If we reverse the orientation of the sphere we get:

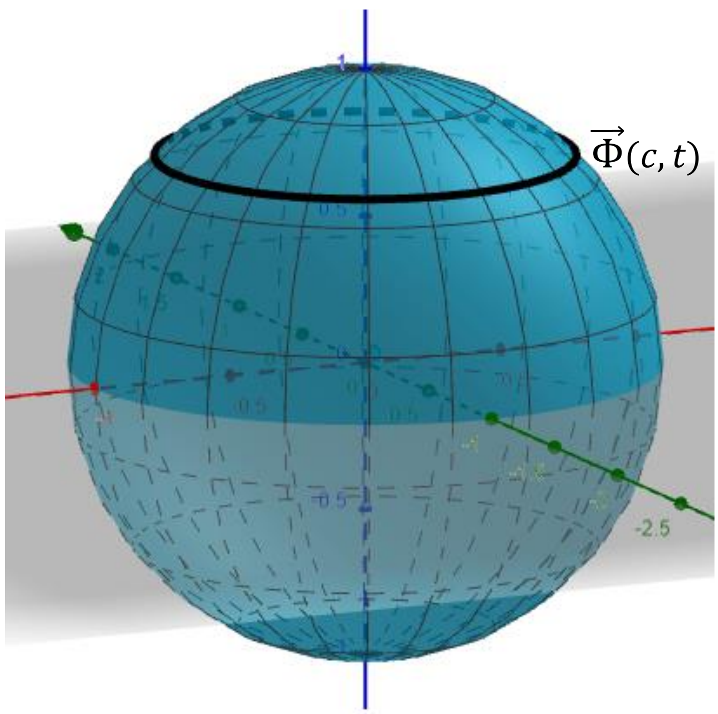
$$\kappa_n = \frac{R(u')^2 + R\sin^2\varphi (v')^2}{R^2(u')^2 + R^2 \sin^2 \varphi (v')^2} = \frac{1}{R}.$$

Ex. Take the unit sphere parametrized by

$$\overrightarrow{\Phi}(u,v) = (cosv(sinu),\ sinv(sinu),\ cosu);$$

where: $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

Now consider the set of circles on this sphere which are the image under $\overrightarrow{\Phi}$ of $u(t) = c,\ v(t) = t,$ where $0 \leq t \leq 2\pi$ and c is a constant with $0 < c < \pi$. Calculate the geodesic curvature, κ_g, the normal curvature, κ_n, and the curvature, κ, at any point on the circles. Show that $\kappa^2 = \kappa_n^2 + \kappa_g^2$.



Let's start with the following formulas:

$$\kappa_g = \frac{\gamma''(t)\cdot\left(\vec{N}\times\gamma'(t)\right)}{\left(E\left(\frac{du}{dt}\right)^2+2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right)+G\left(\frac{dv}{dt}\right)^2\right)^{\frac{3}{2}}}$$

$$\kappa_n = \frac{L\left(\frac{du}{dt}\right)^2+2M\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right)+N\left(\frac{dv}{dt}\right)^2}{E\left(\frac{du}{dt}\right)^2+2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right)+G\left(\frac{dv}{dt}\right)^2}$$

$$\kappa = \frac{\|\gamma''\times\gamma'\|}{\|\gamma'\|^3}.$$

To calculate κ_g we need to know $\gamma', \gamma'', \vec{N}(t), u'(t), v'(t), E, F,$ and G.

$\gamma(t)$ is the image of $\alpha(t) = \big(u(t), v(t)\big) = (c, t)$ under $\vec{\Phi}$.

$$\gamma(t) = \vec{\Phi}(c,t) = (cost(sinc),\ sint(sinc),\ cosc)$$

$$\gamma'(t) = (-(sinc)sint, (sinc)cost, 0)$$

$$\gamma''(t) = (-(sinc)cost, -(sinc)sint, 0).$$

$\vec{N}(t)$ is the unit normal on the sphere at $\gamma(t)$. Recall that:

$$\vec{N} = \frac{(\vec{\Phi}_u\times\vec{\Phi}_v)}{\|\vec{\Phi}_u\times\vec{\Phi}_v\|}.$$

We saw in an earlier calculation that for the unit sphere this becomes:

$$\vec{N}(u,v) = (cosv(sinu),\ sinv(sinu),\ cosu).$$

So when $u(t) = c,\ v(t) = t$ we get:

$$\vec{N}(t) = (cost(sinc),\ sint(sinc),\ cosc)$$

$$\vec{N}(t) \times \gamma'(t) = (-(sinc)(cosc)cost, -(sinc)(cosc)sint, \sin^2 c)\ .$$

(the above comes from a straight forward calculation of $\vec{N}(t) \times \gamma'(t)$).

Now dot this result with $\gamma''(t)$ to get:

$$\gamma''(t) \cdot \left(\vec{N}(t) \times \gamma'(t)\right)$$

$$= (-(sinc)cost, -(sinc)sint, 0)$$

$$\cdot\ (-(sinc)(cosc)cost, -(sinc)(cosc)sint, \sin^2 c)$$

$$= (\sin^2 c)(cosc).$$

We also know from an earlier calculation that for this parametrization of the unit sphere the first fundamental form is:

$$\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$$

so $E = 1,\ F = 0,\ G = \sin^2 u = \sin^2 c.$

Finally, $u(t) = c$ so $u'(t) = 0$

$v(t) = t$ so $v'(t) = 1.$

Plugging into the formula for κ_g we get:

$$\kappa_g = \frac{\gamma''(t)\cdot\left(\vec{N}\times\gamma'(t)\right)}{\left(E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2\right)^{\frac{3}{2}}} = \frac{(\sin^2 c)cosc}{(sin^2 c)^{\frac{3}{2}}} = \cot(c)\ .$$

To calculate the normal curvature, κ_n, we need u', v', E, F, G, which we have already calculated, as well as, L, M, N.

However, we know from the previous example that $\kappa_n = \pm 1$ (-1 for this parametrization), depending on which direction we take for the unit normal. In either case, $\kappa_n^2 = 1$.

Finally, to calculate the curvature, κ, we need to find $\|\gamma'' \times \gamma'\|$, and $\|\gamma'\|$. We calculated earlier that:

$$\gamma'(t) = (-(sinc)sint, (sinc)cost, 0)$$

$$\gamma''(t) = (-(sinc)cost, -(sinc)sint, 0).$$

So $\gamma'' \times \gamma' = (\sin^2 c)\vec{k}$ and thus $\|\gamma'' \times \gamma'\| = \sin^2 c$.

$\|\gamma'\| = \sqrt{(\sin^2 c)\sin^2 t + (\sin^2 c)\cos^2 t} = sinc$.

Thus we have:

$\kappa = \dfrac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3} = \dfrac{\sin^2 c}{\sin^3 c} = \csc(c)$ (notice this is 1/(radius of circle))

So $\kappa = \csc(c)$, $\kappa_n = \pm 1$, and $\kappa_g = \cot(c)$ and we have:

$$\kappa_n^2 + \kappa_g^2 = 1 + \cot^2 c = \csc^2 c = \kappa^2 .$$