The Method of Elimination

Given a differential equation:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = f(t)$$

we can write this as:

$$a_n D^n(x) + a_{n-1} D^{n-1}(x) + \cdots + a_1 D(x) + a_0(x) = f(t)$$

where $D(x) = \frac{dx}{dt}$ and $D^n(x) = \frac{d^n x}{dt^n}$.

So we can think of:

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

as acting on a function $x(t)$. Thus, we can write our original differential equation as $Lx = f(t)$.

We call L a **differential operator**. We define multiplication of operators by:

$$L_1 L_2(x) = L_1\big(L_2(x)\big).$$

For example, suppose we have a, b real numbers and

$$L_1 = D + a, \qquad L_2 = D - b.$$

Then $\quad L_1 L_2(x) = (D + a)[(D - b)(x)]$

$$= (D + a)(Dx - bx)$$

$$= D^2 x + aDx - bDx - abx$$

$$= D^2 x + (a - b)Dx - abx = [D^2 + (a - b)D - ab]x.$$

Notice that if the coefficients of L_1 and L_2 are constants we get $L_1 L_2$ by polynomial multiplication and that $L_1 L_2 = L_2 L_1$.

We will now see how solving linear systems of differential equations with constant coefficients is similar to solving systems of linear equations that you studied in Algebra classes.

Recall that if we wanted to solve:

$$2x - 3y = 2$$
$$-3x + 4y = -4$$

We could multiply the first equation by -3 (the coefficient of x in the second equation) and the second equation by 2 (the coefficient of x in the first equation) and subtract the first equation from the second equation to eliminate the variable x.

$$-6x + 8y = -8 \qquad \text{(2 times second equation)}$$
$$\underline{-6x + 9y = -6} \qquad \text{(-3 times the first equation)}$$
$$-y = -2 \quad \Rightarrow \quad y = 2.$$

We can then substitute $y = 2$ in either original equation to get $x = 4$.

We will take a similar approach to solving a linear system of differential equations with constant coefficients. We can write that system as:

$$L_1 x + L_2 y = f_1(t) \qquad (*)$$
$$L_3 x + L_4 y = f_2(t).$$

Ex. Write the following linear system in terms of $(*)$:

$$x' = -3x - 4y$$
$$y' = -2x + y$$

Start by bringing all terms to one side of the equation:

$$x' + 3x + 4y = 0$$
$$2x + y' - y = 0$$

or
$$(D + 3)(x) + 4y = 0$$
$$2x + (D - 1)y = 0.$$

So if we let:

$$L_1 = D + 3 \qquad L_2 = 4$$
$$L_3 = 2 \qquad L_4 = D - 1, \qquad \text{and } f_1(t) = f_2(t) = 0$$

Our system of equations has the same form as $(*)$.

To solve the system of equations given by $(*)$ we multiply the first equation by L_3 and the second equation by L_1 (similar to our approach for algebraic equations).

$$L_3 L_1 x + L_3 L_2 y = L_3 f_1(t)$$
$$L_1 L_3 x + L_1 L_4 y = L_1 f_2(t).$$

Since all of the $L_i{}'s$ have constant coefficients $L_3 L_1 x - L_1 L_3 x = 0.$

So subtracting the first equation from the second gives us:

$$(L_1L_4 - L_3L_2)y = L_1f_2 - L_3f_1.$$

Now we have a differential equation with only one unknown function $y(t)$. We can write this resulting equation as:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix}.$$

Similarly, we could have multiplied the first equation in $(*)$ by L_4 and the second equation by L_2 and subtracted to get:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix}.$$

Notice that if $f_1(t) = f_2(t) = 0$, the right hand side in both cases is 0, and the left hand side gives identical differential equations in x and y.

The determinant expressions for the differential equations are reminiscent of Cramer's Rule for solving linear algebraic equations.

Ex. Find the general solution to the system of equations given by:

$$x' = -3x - 4y$$

$$y' = -2x + \quad y.$$

As we saw in an earlier example:

$$x' + 3x + 4y = 0 \qquad\qquad (**)$$

$$2x + \; y' - \; y = 0$$

or $$(D+3)(x) + \quad 4y = 0$$
$$2x + \quad (D-1)y = 0.$$

So if we let:

$$L_1 = D + 3 \qquad L_2 = 4$$
$$L_3 = 2 \qquad L_4 = D - 1, \qquad \text{and } f_1(t) = f_2(t) = 0.$$

Since $f_1(t) = f_2(t) = 0$ we know:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix} = 0.$$

$$[L_1 L_4 - L_2 L_3]x = [(D+3)(D-1) - (-2)(4)]x = 0$$
$$[D^2 + 2D + 5]x = 0$$
$$x'' + 2x' + 5x = 0.$$

The characteristic equation for $x'' + 2x' + 5x = 0$ is:

$$r^2 + 2r + 5 = 0$$
$$r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i.$$

So the general solution to $x'' + 2x' + 5x = 0$ is:

$$x(t) = e^{-t}(A\cos(2t) + Bsin(2t)).$$

Notice that:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = 0$$

gives us the same differential equation in y that we had in x:

$$y'' + 2y' + 5y = 0.$$

Thus the general solution is:

$$y(t) = e^{-t}(Ccos(2t) + Esin(2t))$$

where, in general, C and E do not equal A and B.

So far our general solution for this linear system is:

$$x(t) = e^{-t}(A\cos(2t) + Bsin(2t))$$

$$y(t) = e^{-t}(Ccos(2t) + Esin(2t)).$$

Although it looks like there are 4 arbitrary constants $A, B, C,$ and E, in fact, there are only two. We can see this by plugging the general solutions for $x(t)$ and $y(t)$ back into either one of the original differential equations.

Differentiating our expression for $x(t)$ we get:

$$x'(t) = e^{-t}\left(-2Asin(2t) + 2Bcos(2t)\right) - e^{-t}(Acos(2t) + Bsin(2t)).$$

Plugging into $x' + 3x + 4y = 0$ $(**)$ we get:

$$0 = e^{-t}\left(-2Asin(2t) + 2Bcos(2t)\right) - e^{-t}\left(Acos(2t) + Bsin(2t)\right)$$
$$+3\left(e^{-t}\left(A\cos(2t) + Bsin(2t)\right)\right) + 4\left(e^{-t}\left(Ccos(2t) + Esin(t2)\right)\right)$$
$$= e^{-t}[(2A + 2B + 4C)\cos(2t) + (-2A + 2B + 4E)\sin(2t)].$$

Since $\cos(t)$ and $\sin(t)$ are linearly independent we have:

$$2A + 2B + 4C = 0 \quad \Rightarrow \qquad C = -\frac{1}{2}(A + B)$$
$$-2A + 2B + 4E = 0 \quad \Rightarrow \qquad E = \frac{1}{2}(A - B).$$

Thus the constant C and E are functions of A and B, i.e., they are not arbitrary constants.

This says that the general solution to our system of equations is:

$$x(t) = e^{-t}(A\cos(2t) + Bsin(2t))$$
$$y(t) = -\frac{1}{2}e^{-t}((A + B)cos(2t) + (B - A)sin(2t)).$$

Ex. Solve the initial value problem:

$$x' = -3x + 2y; \qquad x(0) = 0$$
$$y' = -3x + 4y; \qquad y(0) = 2.$$

We start by putting the equations in the form of $(*)$.

$$x' + 3x - 2y = 0$$
$$3x + y' - 4y = 0.$$

Thus we have:

$$L_1 = D + 3 \qquad L_2 = -2$$
$$L_3 = 3 \qquad L_4 = D - 4; \qquad f_1(t) = f_2(t) = 0.$$

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = 0 \ ; \qquad \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = 0\,.$$

$$[L_1L_4 - L_2L_3]x = [(D+3)(D-4) - (-2)(3)]x = 0$$
$$[D^2 - D - 6]x = 0$$
$$x'' - x' - 6x = 0.$$

The characteristic equation for $x'' - x' - 6x = 0$ is:

$$r^2 - r - 6 = 0$$
$$(r-3)(r+2) = 0$$
$$r = 3, -2.$$

Since $f_1(t) = f_2(t) = 0$ we get the same differential equation in y that we get in x: $y'' - y' - 6y = 0.$

Thus our general solutions are:

$$x(t) = Ae^{3t} + Be^{-2t}$$

$$y(t) = Ce^{3t} + Ee^{-2t}.$$

Next we find the relationships between A, B, C, and E by plugging expressions for $x(t), x'(t)$ and $y(t)$ into $x' + 3x - 2y = 0$.

$x'(t) = 3Ae^{3t} - 2Be^{-2t}$, so we get:

$$0 = x' + 3x - 2y$$

$$0 = 3Ae^{3t} - 2Be^{-2t} + 3(Ae^{3t} + Be^{-2t}) - 2(Ce^{3t} + Ee^{-2t}).$$

$$0 = (6A - 2C)e^{3t} + (B - 2E)e^{2t} .$$

Since e^{3t} and e^{-2t} are linearly independent we have

$$6A - 2C = 0 \quad \Rightarrow \quad C = 3A$$

$$B - 2E = 0 \quad \Rightarrow \quad E = \frac{1}{2}B.$$

Thus the general solution to this system of equations is:

$$x(t) = Ae^{3t} + Be^{-2t}$$

$$y(t) = 3Ae^{3t} + \frac{1}{2}Be^{-2t}.$$

Now we can use our initial conditions to find A and B.

$$0 = x(0) = A + B$$

$$2 = y(0) = 3A + \frac{1}{2}B.$$

Solving these equations we get: $A = \frac{4}{5},\ \ B = -\frac{4}{5}.$

So the solution to our initial value problem is:

$$x(t) = \frac{4}{5}(e^{3t} - e^{-2t})$$

$$y(t) = \frac{2}{5}(6e^{3t} - e^{-2t}).$$

Finally, we do an example where $f_1(t)$ and $f_2(t)$ are not the zero function.

Ex. Find the general solution to the system given by:

$$x' = 2x - 3y + 2\sin(2t)$$

$$y' = x - 2y - \cos(2t).$$

First we rewrite the equations in terms of:

$$L_1x + L_2y = f_1(t)$$

$$L_3x + L_4y = f_2(t).$$

$$x' - 2x + 3y = 2\sin(2t)$$

$$-x + y' + 2y = -\cos(2t).$$

$$L_1 = D - 2 \qquad L_2 = 3, \qquad f_1(t) = 2\sin(2t)$$

$$L_3 = -1 \qquad L_4 = D + 2, \qquad f_2(t) = -\cos(2t).$$

Since $f_1(t)$ and $f_2(t)$ are not zero functions we have:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix}.$$

Starting with the equations in $x(t)$ we get:

$$(L_1L_4 - L_2L_3)x = L_4 f_1 - L_2 f_2$$

$$[(D-2)(D+2) - 3(-1)]x = (D+2)(2\sin(2t)) - 3(-\cos(2t))$$

$$(D^2 - 1)x = 4\cos(2t) + 4\sin(2t) + 3\cos(2t)$$

$$(D^2 - 1)x = 7\cos(2t) + 4\sin(2t).$$

So we must solve: $x'' - x = 7\cos(2t) + 4\sin(2t).$

Solving the homogenous equation we get:

$$x'' - x = 0$$

$$r^2 - 1 = 0 \quad \Rightarrow \quad r = \pm 1.$$

$$x_c(t) = Ae^{-t} + Be^{t}.$$

To find a particular solution we try $x_p = Ecos(2t) + Fsin(2t)$

$$x_p' = -2Esin(2t) + 2Fcos(2t)$$

$$x_p'' = -4Ecos(2t) - 4Fsin(2t).$$

Plugging into $x'' - x = 7\cos(2t) + 4\sin(2t)$ we get:

$$-4Ecos(2t) - 4Fsin(2t) - Ecos(2t) - Fsin(2t) = 7\cos(2t) + 4\sin(2t)$$

$$-5E\cos(2t) - 5Fsin(2t) = 7\cos(2t) + 4\sin(2t)$$

Thus we get: $E = -\frac{7}{5}$, $F = -\frac{4}{5}$.

So the general solution to $x'' - x = 7\cos(2t) + 4\sin(2t)$ is:

$$x(t) = Ae^{-t} + Be^{t} - \frac{1}{5}[7\cos(2t) + 4\sin(2t)].$$

Now we repeat the process for $\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix}$.

The left hand side will give us the same (homogenous) differential equation as we had in x, but the right hand side is different.

$$(L_1L_4 - L_2L_3)y = L_1f_2 - L_3f_1$$

$$(D^2 - 1)y = (D - 2)(-\cos(2t)) - (-1)(2\sin\,(2t)$$

$$y'' - y = 2\sin(2t) + 2\cos(2t) + 2\sin(2t)$$

$$y'' - y = 4\sin(2t) + 2\cos(2t).$$

$$y_c(t) = me^{-t} + ne^{t}$$

For a particular solution we try: $y_p = Ecos(2t) + Fsin(2t)$

$$y_p' = -2Esin(2t) + 2Fcos(2t)$$

$$y_p'' = -4Ecos(2t) - 4Fsin(2t).$$

Plugging into $y'' - y = 4\sin(2t) + 2\cos(2t)$ we get:

$$-4E\cos(2t) - 4Fsin(2t) - Ecos(2t) - Fsin(2t) = 4\sin(2t) + 2\cos(2t)$$

$$-5Ecos(2t) - 5Fsin(2t) = 4\sin(2t) + 2\cos(2t)$$

$$\Rightarrow \qquad E = -\frac{2}{5}, \quad F = -\frac{4}{5}\ .$$

Thus we have: $y(t) = me^{-t} + ne^{t} - \frac{1}{5}[2\cos(2t) + 4\sin(2t)]$.

To find the relationships between $A, B, m,$ and n, we plug expressions for $x(t), x'(t),$ and $y(t)$ into $0 = x' - 2x + 3y - 2\sin(2t)$.

$$x'(t) = -Ae^{-t} + Be^{t} + \frac{14}{5}\sin(2t) - \frac{8}{5}\cos(2t).$$

$$0 = x' - 2x + 3y - 2\sin(2t)$$

$$= -Ae^{-t} + Be^{t} + \frac{14}{5}\sin(2t) - \frac{8}{5}\cos(2t)$$

$$-2(Ae^{-t} + Be^{t} - \frac{1}{5}[7\cos(2t) + 4\sin(2t)])$$

$$3(me^{-t} + ne^{t} - \frac{1}{5}[2\cos(2t) + 4\sin(2t)]) - 2\sin\,(2t).$$

$$0 = (-3A + 3m)e^{-t} + (-B + 3n)e^{t}.$$

Since e^t and e^{-t} are linearly independent:

$$-3A + 3m = 0$$
$$-B + 3n = 0\ .$$

$$\Rightarrow \quad m = A,\ \ n = \frac{B}{3}$$

So the general solution to this system of equations is given by:

$$x(t) = Ae^{-t} + Be^t - \frac{1}{5}[7\cos(2t) + 4\sin(2t)]$$
$$y(t) = Ae^{-t} + \frac{B}{3}e^t - \frac{1}{5}[2\cos(2t) + 4\sin(2t)].$$

Note: we could have just as easily solved for the constants A and B in terms of m and n and written all of the constants in the solution in terms of m and n instead of A and B.