Tensors- HW Problems

1. Suppose the dimension of V is 3 and $\{v_1, v_2, v_3\}$ is a basis for V. Let $T \in \mathfrak{T}^2(V)$ and $w_1 = a_{11}v_1 + a_{12}v_2 + a_{13}v_3$, $w_2 = a_{21}v_1 + a_{22}v_2 + a_{23}v_3$. Show that:

   a. $T(w_1, w_2) = \sum_{i,j=1}^{i,j=3} a_{1i}a_{2j}T(v_i, v_j)$

   b. If $\varphi_i(v_j) = \delta_{ij}$, then $\varphi_1 \otimes \varphi_3(w_1, w_2) = a_{11}a_{23}$.

2. Suppose V is the vector space $\mathbb{R}^2$, and $S: V \times V \to \mathbb{R}$, by $S(v_1, v_2) = v_1 \cdot v_2$ (i.e., the dot product of v_1 and v_2) and $T: V \times V \to \mathbb{R}$, by $T(v_1, v_2) = \det\binom{v_1}{v_2}$. Let $w_1 = (1,2)$, $w_2 = (1, -1)$, $w_3 = (2,1)$, and $w_4 = (-1, -2)$. Evaluate $(S \otimes T)(w_1, w_2, w_3, w_4)$.

3. Suppose that $\{v_1, v_2, v_3\}$ is a basis for V and $\varphi_i(v_j) = \delta_{ij}$. Let $w_1 = 2v_1 + v_2$, $w_2 = v_1 - 2v_3$, $w_3 = 2v_1 - v_2 + v_3$, and $T = \varphi_1 \otimes \varphi_1 \otimes \varphi_3$. Calculate $Alt\, T(w_1, w_2, w_3)$.

4. Let $f: \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation represented in the standard basis $\{e_i\}$ by $f = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & -2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 3 & 0 \end{bmatrix}$. Let $\{\varphi_i\}$ be the dual basis so $\varphi_i(e_j) = \delta_{ij}$. Now let $T \in \mathfrak{J}^2(\mathbb{R}^4)$ given by

$T = \varphi_2 \otimes \varphi_3 - \varphi_1 \otimes \varphi_4$.
Find $f^*T(w_1, w_2)$ where
$w_1 = (1, -1, 1, 2)$ and $w_2 = (-2, 2, 0, 1)$.

5. Let $T \in \mathfrak{J}^2(\mathbb{R}^2)$ be the dot product of two vectors, i.e., if

$v_1 = a_1 e_1 + b_1 e_2$, and $v_2 = a_2 e_1 + b_2 e_2$, then
$T(v_1, v_2) = a_1 a_2 + b_1 b_2$. Suppose $\varphi_i(e_j) = \delta_{ij}$.
We know that $\{\varphi_1 \otimes \varphi_1, \varphi_1 \otimes \varphi_2, \varphi_2 \otimes \varphi_1, \varphi_2 \otimes \varphi_2\}$ is a basis for $\mathfrak{J}^2(\mathbb{R}^2)$. Write T as a linear combination of this basis.
Hint: $a_1 a_2 + b_1 b_2 = T(v_1, v_2) =$
$\lambda_1(\varphi_1 \otimes \varphi_1)(v_1, v_2) + \lambda_2(\varphi_1 \otimes \varphi_2)(v_1, v_2)$
$+\lambda_3(\varphi_2 \otimes \varphi_1)(v_1, v_2) + \lambda_4(\varphi_2 \otimes \varphi_2)(v_1, v_2)$.
Now plug in v_1 and v_2 into the RHS and calculate.