# The First Fundamental Form of a Surface in $\mathbb{R}^3$

We know from multivariable calculus that if $\gamma(t) = \big(x(t), y(t), z(t)\big)$ is a curve in $\mathbb{R}^3$, then it's length is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt\ .$$

$\gamma'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ and $\|\gamma'(t)\|^2 = \gamma' \cdot \gamma' = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$

so we can also write:

$$L = \int_a^b \|\gamma'(t)\|\ dt.$$

Now we want to consider a formula for the length of a curve that lies on a smooth surface $S \subseteq \mathbb{R}^3$. Suppose $S$ is parameterized by:

$$\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$$

$$\vec{\Phi}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big).$$

If $S$ is a regular surface (i.e. $\vec{\Phi}_u\ \times\ \vec{\Phi}_v \neq \vec{0}$), then $\vec{\Phi}_u$ and $\vec{\Phi}_v$ span the tangent space of $S$ at $p \in S$. Thus, any vector $\vec{w}$ in $T_pS$ can be written as:

$$\vec{w} = a\vec{\Phi}_u + b\vec{\Phi}_v\ ; \quad a, b \in \mathbb{R}.$$

Notice that:

$$\begin{aligned}\vec{w} \cdot \vec{w} &= \left(a\vec{\Phi}_u + b\vec{\Phi}_v\right) \cdot \left(a\vec{\Phi}_u + b\vec{\Phi}_v\right) \\ &= a^2\vec{\Phi}_u \cdot \vec{\Phi}_u + 2ab\vec{\Phi}_u \cdot \vec{\Phi}_v + b^2\vec{\Phi}_v \cdot \vec{\Phi}_v\ .\end{aligned}$$

If we let:

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u$$
$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = \vec{\Phi}_v \cdot \vec{\Phi}_u$$
$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v$$

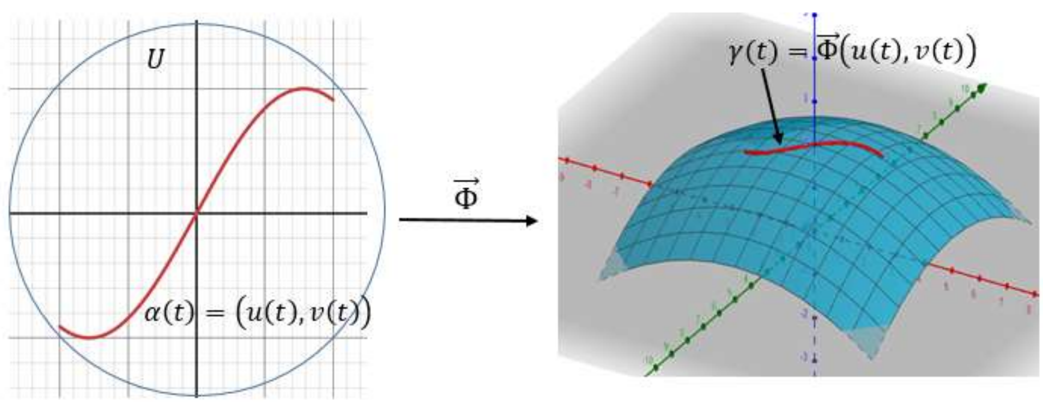
then:

$$\vec{w} \cdot \vec{w} = Ea^2 + 2Fab + Gb^2.$$

In fact, the matrix: $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ represents a bilinear form on $T_pS$. This bilinear form is called the **first fundamental form** on $S$. If $\vec{v} = c\vec{\Phi}_u + d\vec{\Phi}_v$ then

$$\vec{w} \cdot \vec{v} = (a \quad b)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = acE + (ad + bc)F + bdG.$$

Suppose that $\gamma$ is a curve on $S$ such that: $\gamma(t) = \vec{\Phi}\big(u(t), v(t)\big)$.




By the chain rule:

$$\gamma'(t) = \vec{\Phi}_u u'(t) + \vec{\Phi}_v v'(t)$$

Thus we have:

$$\begin{aligned} \|\gamma'(t)\|^2 &= \gamma'(t) \cdot \gamma'(t) = \big(u'\vec{\Phi}_u + v'\vec{\Phi}_v\big) \cdot \big(u'\vec{\Phi}_u + v'\vec{\Phi}_v\big) \\ &= (u')^2\vec{\Phi}_u \cdot \vec{\Phi}_u + 2(u')(v')\vec{\Phi}_u \cdot \vec{\Phi}_v + (v')^2\vec{\Phi}_v \cdot \vec{\Phi}_v \\ &= E(u')^2 + 2F(u')(v') + G(v')^2. \end{aligned}$$

And so we get:

$$L = \int_a^b \|\gamma'(t)\| \; dt = \int_a^b (E(u')^2 + 2F(u')(v') + G(v')^2)^{\frac{1}{2}} \; dt.$$

Ex. Let $\overrightarrow{\Phi}(u,v) = \big((\cos v)\sin u\,,(\sin v)\sin u\,,\cos u\big)$ be a parameterization of the unit sphere. Find the first fundamental form at $\overrightarrow{\Phi}(u,v)$ and use it to find the length of the portion of the great circle starting at $(0,0,1)$ and ending at $(1,0,0)$.

$$\overrightarrow{\Phi}_u = ((\cos v)\cos u\,,(\sin v)\cos u\,,-\sin u)$$
$$\overrightarrow{\Phi}_v = (-(\sin v)\sin u\,,(\cos v)\sin u\,,0)$$

$$\begin{aligned} E &= \overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_u \\ &= ((\cos v)\cos u\,,(\sin v)\cos u\,,-\sin u) \\ &\qquad \cdot ((\cos v)\cos u\,,(\sin v)\cos u\,,-\sin u) \\ &= (\cos^2 v)\cos^2 u + (\sin^2 v)\cos^2 u + \sin^2 u \\ &= \cos^2 u + \sin^2 u \\ &= 1 \end{aligned}$$

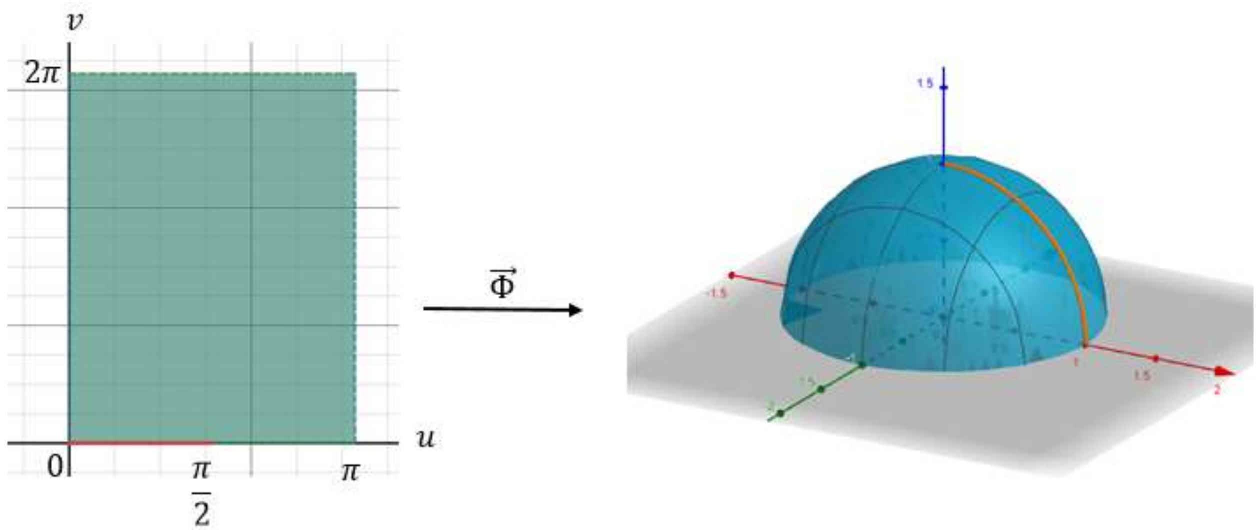
$$\begin{aligned} F &= \overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_v \\ &= ((\cos v)\cos u\,,(\sin v)\cos u\,,-\sin u) \\ &\qquad \cdot (-(\sin v)\sin u\,,(\cos v)\sin u\,,0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} G &= \overrightarrow{\Phi}_v \cdot \overrightarrow{\Phi}_v \\ &= (-(\sin v)\sin u\,,(\cos v)\sin u\,,0) \\ &\qquad \cdot (-(\sin v)\sin u\,,(\cos v)\sin u\,,0) \\ &= (\sin^2 v)\sin^2 u + (\cos^2 v)\sin^2 u \\ &= \sin^2 u. \end{aligned}$$

So the first fundamental form of $\overrightarrow{\Phi}$ is:

$$\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}.$$

The portion of the great circle starting at $(0,0,1)$ and ending at $(1,0,0)$ is the image of the line segment starting at $u=0, v=0$ and ending at $u=\frac{\pi}{2}$ and $v=0$.



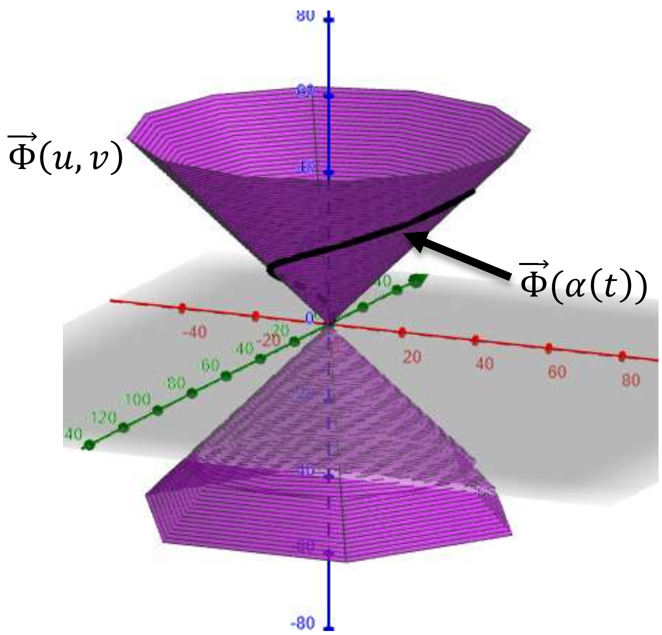

We can parameterize this curve in the $u, v$ plane by:

$$u(t)=t, \quad v(t)=0 \ ; \quad 0 \leq t \leq \frac{\pi}{2}\,.$$

Thus, $\quad u'(t)=1, \;\; v'(t)=0$ and

$$L=\int_0^{\frac{\pi}{2}}(E(u')^2+2F(u')(v')+G(v')^2)^{\frac{1}{2}}\;dt=\int_0^{\frac{\pi}{2}}1\,dt=\frac{\pi}{2}\,.$$

Ex. Find the first fundamental form of $\vec{\Phi}(u,v) = (vcosu, vsinu, v)$; where $0 \le u \le 2\pi$, $v \in \mathbb{R}$, $v \neq 0$, and find the length of the image under $\vec{\Phi}$ of the curve $\alpha(t) = (t, t^2)$; where $0 \le t \le 2\pi$.




$$\vec{\Phi}(u,v) = (vcosu, vsinu, v)$$
$$\vec{\Phi}_u = (-\mathrm{v}\sin u\,, v\cos u\,, 0), \qquad \vec{\Phi}_v = (cosu, sinu, 1).$$

$$E = \vec{\Phi}_u \cdot \vec{\Phi}_u = (-\mathrm{v}\sin u\,, v\cos u\,, 0) \cdot (-v\sin u\,, v\cos u\,, 0)$$
$$= v^2 \sin^2 u + v^2 \cos^2 u = v^2$$

$$F = \vec{\Phi}_u \cdot \vec{\Phi}_v = (-\mathrm{v}\sin u\,, \mathrm{v}\cos u\,, 0) \cdot (cosu, sinu, 1) = 0$$

$$G = \vec{\Phi}_v \cdot \vec{\Phi}_v = (cosu, sinu, 1) \cdot (cosu, sinu, 1) = 2.$$

So the first fundamental form is given by:

$$g = \begin{pmatrix} v^2 & 0 \\ 0 & 2 \end{pmatrix}.$$

$$L = \int_a^b (E(u')^2 + 2F(u')(v') + G(v')^2)^{\frac{1}{2}}\, dt.$$

$\alpha(t) = \big(u(t), v(t)\big) = (t, t^2)$;

so $u(t) = t, \quad v(t) = t^2 \quad \Longrightarrow \quad u'(t) = 1, \quad v'(t) = 2t$ .

$$\begin{aligned} L(\overrightarrow{\Phi}(\alpha(t))) &= \int_{t=0}^{t=2\pi} (E(u')^2 + G(v')^2)^{\frac{1}{2}}\, dt \\ &= \int_{t=0}^{t=2\pi} ((v^2(u')^2 + 2(v')^2)^{\frac{1}{2}}\, dt \\ &= \int_{t=0}^{t=2\pi} ((t^4(1)^2 + 2(2t)^2)^{\frac{1}{2}}\, dt \\ &= \int_{t=0}^{t=2\pi} (t^4 + 8t^2)^{\frac{1}{2}}\, dt \\ &= \int_{t=0}^{t=2\pi} t(t^2 + 8)^{\frac{1}{2}}\, dt \end{aligned}$$

Let $w = t^2 + 8$; when $t = 0, \quad w = 8$

$dw = 2tdt$ when $t = 2\pi, \quad w = 4\pi^2 + 8$

$\frac{1}{2} dw = tdt$

$$\begin{aligned} L\big(\overrightarrow{\Phi}(\alpha(t))\big) &= \frac{1}{2} \int_{w=8}^{w=4\pi^2+8} w^{\frac{1}{2}} dw \\ &= \frac{1}{2}\left(\frac{2}{3}\right) w^{\frac{3}{2}} \Big|_{w=8}^{w=4\pi^2+8} \\ &= \frac{1}{3}[(4\pi^2 + 8)^{\frac{3}{2}} - (8)^{\frac{3}{2}}] \\ &= \frac{8}{3}\left[(\pi^2 + 2)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]. \end{aligned}$$

If we let $E = g_{11}, F = g_{12} = g_{21}$, and $G = g_{22}$, then the matrix representing the first fundamental form is given by:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

This is called the **metric tensor** for the parameterization $\overrightarrow{\Phi}$ of the surface $S$. Notice that $g$ is a bilinear form:

$$g: T_pS \times T_pS \to \mathbb{R}$$

where,

$$g(\vec{w}_1, \vec{w}_2) = \vec{w}_1 \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \vec{w}_2.$$

This is a $2$ tensor on the vector space $V = T_pS$. We will discuss tensors later in this course.