Stokes' Theorem on Manifolds

Stokes' Theorem for Compact Oriented Manifolds:

> If M is a compact oriented k-dimensional manifold with boundary and ω is a $(k-1)$-form on M, then
>
> $$\int_M d\omega = \int_{\partial M} \omega.$$

We saw earlier that this theorem is true when M is a k-chain. To get this result for a compact manifold, we take a partition of unity $\{\psi_\alpha\}_{\alpha\in I}$ for M subordinate to an open cover $\mathcal{U}$. So for any point $x \in M$:

$$0 = d(1) = d\left(\sum_{\alpha\in \mathrm{I}} \psi_\alpha\right) = \sum_{\alpha\in \mathrm{I}} d\psi_\alpha$$

Thus

$$\sum_{\alpha\in \mathrm{I}} d\psi_\alpha \wedge \omega = 0.$$

Since M is compact, $\mathcal{U}$ has a finite subcover. The above sum is finite and

$$\sum_{\alpha\in \mathrm{I}} \int_M d\psi_\alpha \wedge \omega = 0.$$

Therefore:

$$\int_M d\omega = \sum_{\alpha\in \mathrm{I}} \int_M \psi_\alpha(d\omega) = \sum_{\alpha\in \mathrm{I}} \int_M (d\psi_\alpha \wedge \omega + \psi_\alpha \wedge d\omega)$$

$$= \sum_{\alpha\in \mathrm{I}} \int_M d(\psi_\alpha\omega) = \sum_{\alpha\in \mathrm{I}} \int_{\partial M} \psi_\alpha\omega = \int_{\partial M} \omega.$$

Corollary 1: If M is a compact manifold without boundary (e.g. a sphere or torus) and ω is a $k-1$ form, then:

$$\int_M d\omega = 0.$$

Proof:

$$\int_M d\omega = \int_{\partial M} \omega$$

But $\partial M = \phi$, so:

$$\int_{\partial M} \omega = 0.$$

Corollary 2: If M is a compact manifold (with or without boundary) and ω is a $k-1$ form that is closed (i.e. $d\omega = 0$), then:

$$\int_{\partial M} \omega = 0.$$

Proof:

$$\int_{\partial M} \omega = \int_M d\omega = \int_M 0 = 0.$$

Notice that if M is not compact, then Stokes' Theorem need not hold. For example, if $M = D - (0,0)$, where $D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ and $\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$, then $d\omega = 0$. But we have:

$$\int_{\partial M} \omega = 2\pi \neq \int_M d\omega = 0.$$

Ex. Let $\omega = (1 + xy^2)e^{xy^2}dx + 2x^2ye^{xy^2}dy$ be a 1-form on $S^1 = \{(x, y) \in \mathbb{R}^2 |\, x^2 + y^2 = 1\}$. Evaluate:

$$\int_{S^1} \omega$$

Notice:

$$d\omega = \left((1 + xy^2)(2xy)e^{xy^2} + (2xy)e^{xy^2}\right) dy \wedge dx + \left(2x^2y(y^2)e^{xy^2} + 4xye^{xy^2}\right) dx \wedge dy = 0.$$

So by Stokes' Theorem:

$$\int_{S^1} \omega = \int_D d\omega = \int_D 0 = 0$$

where D is the unit disk $x^2 + y^2 \leq 1$ and $S^1 = \partial D$.

Ex. Using Stokes' Theorem evaluate:

$$\iint_{S^2} \omega$$

where $\omega = z^2\, dx \wedge dy + \ x\, dy \wedge dz - y\, dx \wedge dz$ and S^2 is the unit sphere.

Notice:

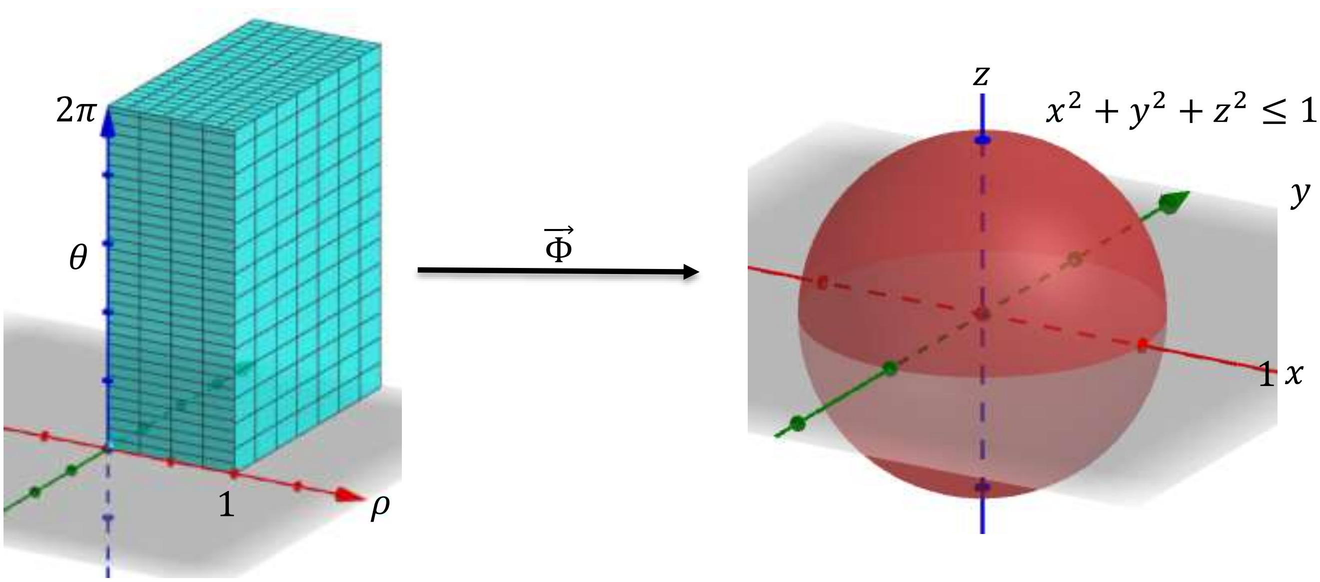
$$\iint_{S^2} \omega = \iiint_B d\omega$$

where B is the unit ball in $\mathbb{R}^3$.

$$d\omega = d(z^2\, dx \wedge dy + \, x\, dy \wedge dz - y\, dx \wedge dz\,)$$
$$= 2z\, dz \wedge dx \wedge dy + \, dx \wedge dy \wedge dz - dy \wedge dx \wedge dz$$
$$= (2z + 2) dx \wedge dy \wedge dz.$$

Now let $\quad \vec{\Phi}\colon [0,1] \times [0,\pi] \times [0,2\pi] \to \mathbb{R}^3$ by:

$$\vec{\Phi}(\rho,\phi,\theta) = (\rho\cos\theta\sin\phi\,, \rho\sin\theta\sin\phi\,, \rho\cos\phi)$$



$$\vec{\Phi}^*((2z+2)dx \wedge dy \wedge dz) = (2\rho\cos\phi + 2)\big(\det \vec{\Phi}'\big) d\rho \wedge d\phi \wedge d\theta$$

$$= (2\rho\cos\phi + 2)(\rho^2 \sin\phi) d\rho \wedge d\phi \wedge d\theta.$$

$$\iint_{S^2} \omega = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} (2\rho^3 \cos\phi \sin\phi + 2\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \frac{\rho^4}{2} \cos\phi \sin\phi + \frac{2\rho^3}{3} \sin\phi \Big|_0^1 \, d\phi\, d\theta$$

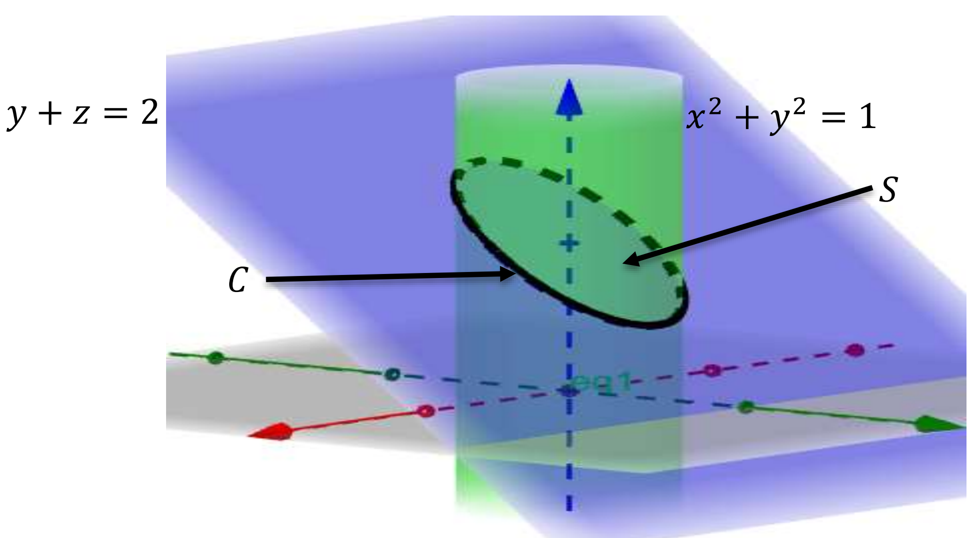
$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \left(\frac{1}{2} \cos\phi \sin\phi + \frac{2}{3} \sin\phi\right) d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{4} \sin^2\phi - \frac{2}{3} \cos\phi \Big|_0^{\pi} \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{4}{3}\, d\theta = \frac{4}{3}\, d\theta \Big|_0^{2\pi} = \frac{8\pi}{3}.$$

Stokes' Theorem can be used at times to simplify calculations.

Ex. Let $\omega = -y^2\, dx + x\, dy + z^2\, dz$. Evaluate $\int_C \omega$ where C is the curve given by the intersection in $\mathbb{R}^3$ of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.



By Stokes' Theorem, where S is any smooth compact surface with $C = \partial S$:

$$\int_C \omega = \int_S d\omega$$

where: $d\omega = d(-y^2\,dx + x\,dy + z^2\,dz) = (1+2y)dx \wedge dy.$

The easiest S to take is the (elliptical) region in the plane,

$$y + z = 2.$$

Notice that the projection into the xy-plane is just the disk, D, $x^2 + y^2 \leq 1$.

If we take $f: D \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ by $f(u,v) = (u, v, 2-v)$:

$$f^*\big((1+2y)dx \wedge dy\big) = (1+2v)du \wedge dv$$

Thus,

$$\int_C -y^2\,dx + x\,dy + z^2\,dz = \iint_D (1+2v)\,du \wedge dv$$

Changing to polar coordinates, we get:

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1 + 2r\sin\theta)\,r\,dr\,d\theta = \int_{\theta=0}^{2\pi} \frac{r^2}{2} + \frac{2r^3}{3}\sin\theta \Big|_0^1 \,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin\theta\right) d\theta = \frac{1}{2}\theta - \frac{2}{3}\cos\theta \Big|_0^{2\pi} = \pi.$$

One could also evaluate the original line integral by parameterizing the curve C by $c(t) = (\cos t, \sin t, 2 - \sin t)$; $0 \leq t \leq 2\pi$ but this calculation is messier.

Ex. Use Stokes' Theorem and valuing the integral directly to evaluate the following given that $\omega = z\, dx \wedge dy + x\, dy \wedge dz + y\, dz \wedge dx$.

$$\iint_{S^2} \omega$$

where S^2 is the unit sphere in $\mathbb{R}^3$.

By Stokes' Theorem:

$$d\omega = dz \wedge dx \wedge dy + dx \wedge dy \wedge dz + dy \wedge dz \wedge dx$$

$$= 3dx \wedge dy \wedge dz.$$

$$\iint_{S^2} \omega = \iiint_B 3dx\, dy\, dz = 3(\text{volume of the unit ball})$$

$$= 3\left(\frac{4}{3}\pi(1)^3\right) = 4\pi.$$

Valuing the integral directly:

$$c: [0, \pi] \times [0, 2\pi] \to S^2 \subseteq \mathbb{R}^3$$

$$c(\phi, \theta) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

$$c^*(\omega) = \cos\phi\, d(c^*x) \wedge d(c^*y) + \cos\theta \sin\phi\, d(c^*y) \wedge d(c^*z)$$
$$+ \sin\theta \sin\phi\, d(c^*z) \wedge d(c^*x)$$

$$= \cos\phi\, d(\cos\theta\sin\phi) \wedge d(\sin\theta\sin\phi)$$
$$+ \cos\theta\sin\phi\, d(\sin\theta\sin\phi) \wedge d(\cos\phi)$$
$$+ \sin\theta\sin\phi\, d(\cos\phi) \wedge d(\cos\theta\sin\phi)$$

$$= \cos\phi\,(-\sin\theta\sin\phi\, d\theta + \cos\theta\cos\phi\, d\phi) \wedge (\cos\theta\sin\phi\, d\theta + \sin\theta\cos\phi) d\phi]$$

$$+ \cos\theta\sin\phi\,[(\cos\theta\sin\phi\, d\theta + \sin\theta\cos\phi\, d\phi) \wedge (-\sin\phi\, d\phi)]$$

$$+ \sin\theta\sin\phi\,[(-\sin\phi\, d\phi) \wedge (-\sin\theta\sin\phi\, d\theta + \cos\theta\cos\phi\, d\phi)]$$

$$= -\cos^2\phi\sin\phi\, d\theta\, d\phi - \cos^2\theta\sin^3\phi\, d\theta\, d\phi - \sin^2\theta\sin^3\phi\, d\theta\, d\phi$$

$$= -[\cos^2\phi\sin\phi\, d\theta\, d\phi + \sin^3\phi\, d\theta\, d\phi] = -[\sin\phi\, d\theta\, d\phi]$$

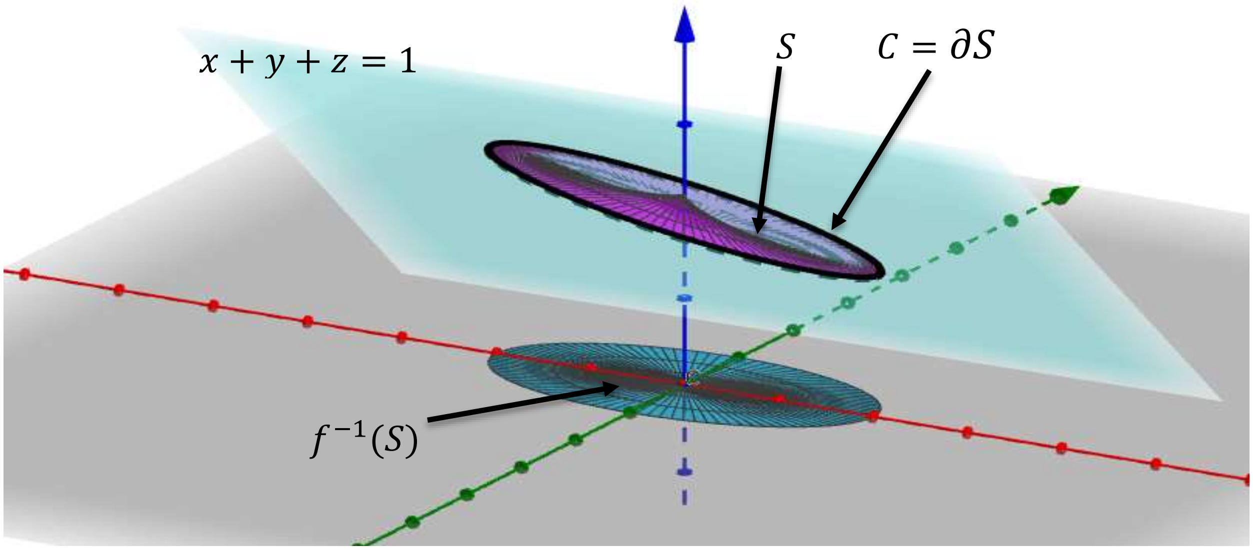
$$= \sin\phi\, d\phi \wedge d\theta.$$

$$\iint_{S^2} \omega = \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} -\cos\phi\Big|_0^{\pi}\, d\theta$$

$$= \int_0^{2\pi} 2\, d\theta = 4\pi.$$

Ex. Let C be a smooth simple closed curve that lies in the plane
$x + y + z = 1$.

Show that: $\int_C (z\,dx - 2x\,dy + 3y\,dz)$ depends only on the area enclosed by C and not the shape or location of C.



By Stokes' Theorem:

$$\int_C (z\,dx - 2x\,dy + 3y\,dz) = \iint_S d(z\,dx - 2x\,dy + 3y\,dz)$$

where $\partial S = C$ and we can take S to lie in the plane $x + y + z = 1$.

$$d(z\,dx - 2x\,dy + 3y\,dz) = dz \wedge dx - 2\,dx \wedge dy + 3\,dy \wedge dz$$

$$\int_C (z\,dx - 2x\,dy + 3y\,dz) = \iint_S dz \wedge dx - 2\,dx \wedge dy + 3\,dy \wedge dz$$

Now let parametrize the plane $x+y+z=1$ by:

$$f:\mathbb{R}^2\to\mathbb{R}^3 \text{ by } f(u,v)=(u,v,1-u-v)$$

$$\iint_S dz\wedge\ dx-2\ dx\wedge\ dy+3\ dy\wedge\ dz$$

$$=\iint_{f^{-1}(S)} f^*(dz\wedge\ dx-2\ dx\wedge\ dy+3\ dy\wedge\ dz)$$

$$f^*(dz\wedge\ dx-2\ dx\wedge\ dy+3\ dy\wedge\ dz)$$
$$=d\big(f^*(z)\big)\wedge d\big(f^*(x)\big)-2d\big(f^*(x)\big)\wedge d\big(f^*(y)\big)$$
$$+3d\big(f^*(y)\big)\wedge d\big(f^*(z)\big)$$

$$=d(1-u-v)\wedge du-2\ du\wedge dv+3dv\wedge d(1-u-v)$$

$$=-2\ du\wedge dv$$

$$\int_C (z\ dx-2x\ dy+3y\ dz)=-\iint_{f^{-1}(S)} 2\ du\ dv$$
$$=-2(area\ of\ f^{-1}(S)).$$

But notice that the surface area of S, (i.e. the area enclosed by C) is given by:

Surface area of $S=\iint_{f^{-1}(S)}\|f_u\times f_v\|dudv$

$f_u=(1,0,-1)$, $f_v=(0,1,-1)$ and $f_u\times f_v=\vec{\imath}+\vec{\jmath}+\vec{k}$,
so $\|f_u\times f_v\|=\sqrt{3}$.

Surface area of $S=\iint_{f^{-1}(S)}\sqrt{3}dudv=\sqrt{3}(area\ of\ f^{-1}(S))$.

Thus we have:

$$\int_C (z\,dx - 2x\,dy + 3y\,dz) = -2(area\ of\ f^{-1}(S))$$
$$= \frac{-2}{\sqrt{3}}(\text{Surface area of } S)$$

So the original integral only depends on the area enclosed by C (which is the surface area of S).