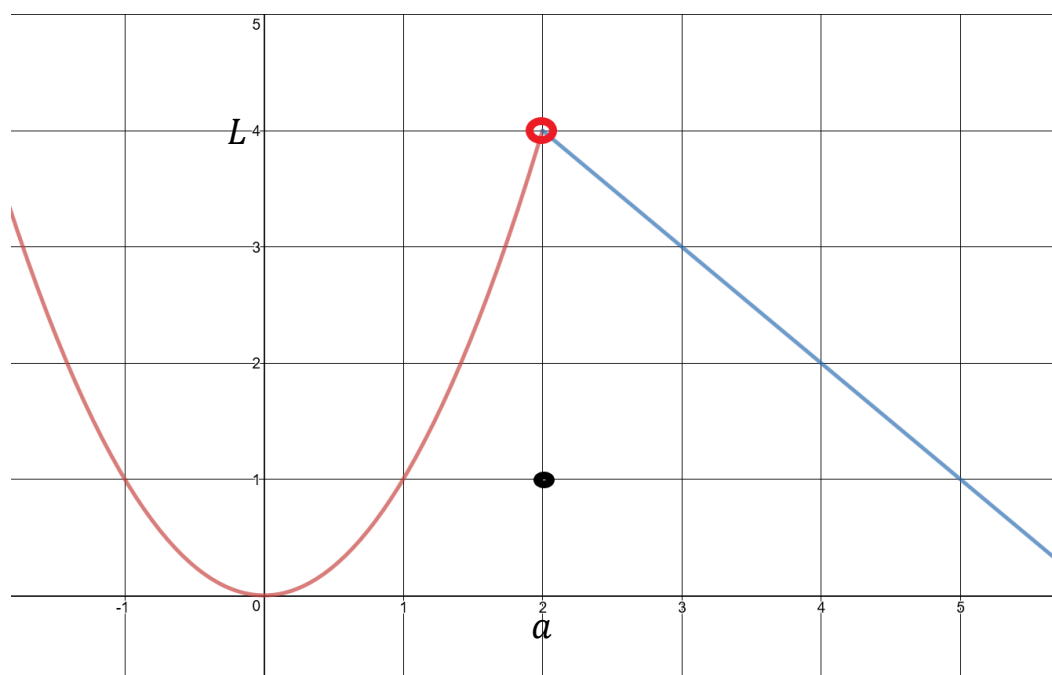
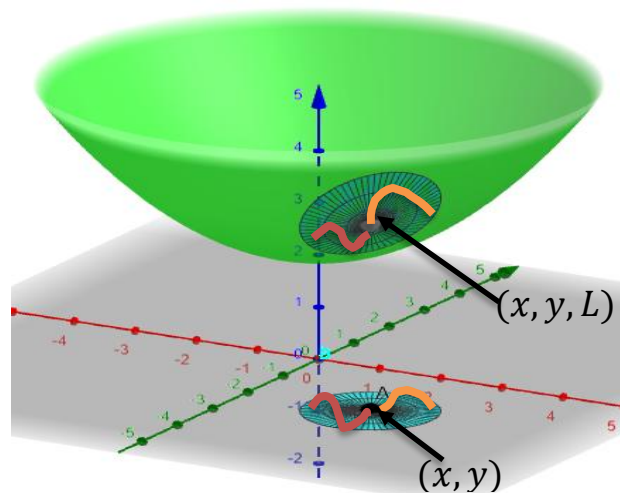


Limits & Continuity

In 1 variable, $\lim_{x \rightarrow a} f(x) = L$ meant as x approaches a from either direction, $f(x)$ approaches L . Notice, we don't care what the value of $f(a)$ is, or even if it's defined at $x = a$. We only care that as x approaches " a " from the left and right, $f(x)$ approaches L .



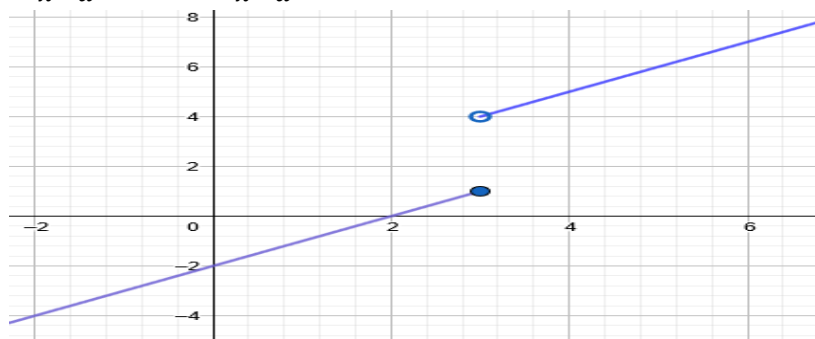
We have a similar meaning for: $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$. As (x,y) approaches (a,b) along any path, $f(x,y)$ approaches L . In 2 dimensions there are many more ways on which (x,y) can approach (a,b) .



Again, we don't care what the value of $f(x, y)$ is at (a, b) , or even if it's defined there, only that as you approach (a, b) along any path in the domain, $f(x, y)$ approaches L .

For 1 variable, to have a limit we need: $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$; only 2 directions.

If $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$, then the limit doesn't exist.



$$\lim_{x \rightarrow 3^+} f(x) \neq \lim_{x \rightarrow 3^-} f(x)$$

For $f(x, y)$, the limit along all paths must exist and be the same number.

Ex. Show $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ doesn't exist.

Notice as $(x, y) \rightarrow (0, 0)$, that $f(x, y) \rightarrow \frac{0}{0}$.

First, let's approach $(0, 0)$ along the x -axis; i.e. along $y = 0$.

$$f(x, 0) = \frac{x^2 - 0^2}{x^2 + 0^2} = 1 \text{ for all } x \neq 0.$$

So as $(x, 0) \rightarrow (0, 0)$, $f(x, y) \rightarrow 1$.

Now approach $(0, 0)$ along the y -axis; i.e. along $x = 0$.

$$f(0, y) = \frac{0^2 - y^2}{0^2 + y^2} = -1 \text{ for all } y \neq 0.$$

So as $(0, y) \rightarrow (0, 0)$, $f(x, y) \rightarrow -1$.

So $f(x, y)$ approaches different values along 2 different paths, hence it doesn't have a limit.

A limit can also fail to exist because the function approaches $\pm\infty$.

Ex. $\lim_{(x,y) \rightarrow (0,0)} \ln(x^2 + y^2) = -\infty$

Ex. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2 + 1}{x^2 + y^2} = +\infty$

If you don't have $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \frac{0}{0}$ or $\frac{k}{0}$; try just plugging in:

Ex. Evaluate $\lim_{(x,y) \rightarrow (1,-2)} \frac{x^2 + y^2 - 1}{xy}$

$$\lim_{(x,y) \rightarrow (1,-2)} \frac{x^2 + y^2 - 1}{xy} = \frac{1^2 + (-2)^2 - 1}{1(-2)} = \frac{1 + 4 - 1}{-2} = \frac{4}{-2} = -2.$$

Ex. Does $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^4 + y^4}$ exist?

Notice if $x = 0$: $f(0, y) = \frac{0}{0 + y^4} = 0 \rightarrow 0$ as $(0, y) \rightarrow (0, 0)$

$y = 0$: $f(x, 0) = \frac{0}{x^4 + 0} = 0 \rightarrow 0$ as $(x, 0) \rightarrow (0, 0)$.

What about approaching $(0,0)$ by a non-vertical or horizontal line:
for example; $y = mx$; $m \neq 0$?

$$f(x, mx) = \frac{x^2(m^2 x^2)}{x^4 + m^4 x^4} = \frac{x^4(m^2)}{x^4(1 + m^4)} = \frac{m^2}{1 + m^4} \neq 0, \text{ when } m \neq 0.$$

so: $f(x, mx) \rightarrow \frac{m^2}{1 + m^4}$ as $(x, mx) \rightarrow (0, 0)$; so the limit does not exist.

Ex. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4+y^2}$ doesn't exist.

Along $x = 0$: $f(0, y) = \frac{2(0)^2y}{0^4+y^2} = 0$ so $f(0, y) \rightarrow 0$.

Along $y = 0$: $f(x, 0) = \frac{2(x^2)0}{x^4+0^2} = 0$ so $f(x, 0) \rightarrow 0$.

Along $y = mx$:
$$f(x, mx) = \frac{2x^2(mx)}{x^4+(mx)^2} = \frac{2mx^3}{x^4+m^2x^2} = \frac{2mx^3}{x^2(x^2+m^2)}$$
$$= \frac{2mx}{x^2+m^2} \rightarrow 0.$$

Along $y = x^2$: $f(x, x^2) = \frac{2x^2(x^2)}{x^4+(x^2)^2} = \frac{2x^4}{2x^4} = 1.$

So $f(x, x^2) \rightarrow 1$ as $(x, x^2) \rightarrow (0,0)$, thus the limit does not exist.

Note: You can NEVER prove that $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ by showing that the limit is the same along any finite number of paths. You can only show that the limit doesn't exist by testing a finite number of paths and showing that the limit is different along 2 of them.

Just because you have $f(x, y) \rightarrow \frac{0}{0}$ doesn't mean there is no limit.

$$\text{Ex. } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2 + x^2}{x^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2(y^2 + 1)}{x^2} = 1$$

Ex. Use polar coordinates to evaluate $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$.

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} &= \lim_{r \rightarrow 0^+} \frac{\sin(r^2)}{r^2}; \quad \text{Now using L'hospital's rule we get} \\ &= \lim_{r \rightarrow 0^+} \frac{(\cos r^2) 2r}{2r} = 1. \end{aligned}$$

Continuity

In 1 variable, $f(x)$ is continuous at $x = a$ means:

$$\lim_{x \rightarrow a} f(x) = f(a).$$

We have a similar definition for 2 variables:

Def. $f(x, y)$ is **continuous at (a, b)** if:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b).$$

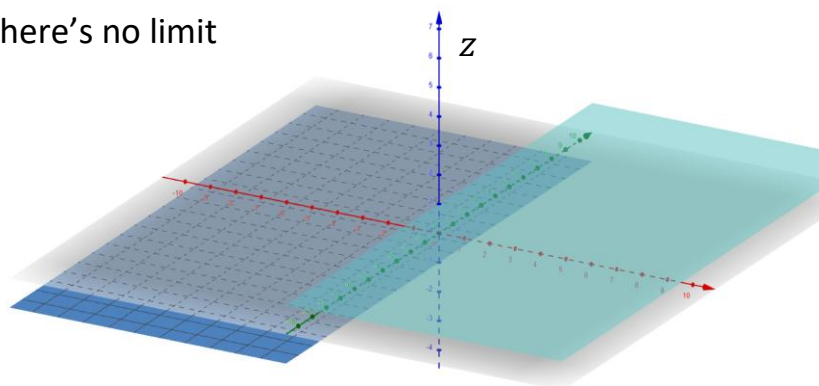
We say f is **continuous on D** if f is continuous at every point (a, b) in D .

$f(x, y)$ can fail to be continuous at (a, b) if:

1. $f(x, y)$ has no limit at (a, b) , for example:

a. $f(x, y) = \frac{1}{xy}$ at $(0,0)$ there's no limit

b. $f(x, y) = 1 \quad x > 0$
 $\quad \quad \quad = -1 \quad x \leq 0$
 half planes



2. $f(x, y)$ has a limit but the function's value is not equal to it there

$$\begin{aligned} f(x, y) &= x^2 + y^2 & x \neq 0, \quad y \neq 0 \\ &= 3 & x = 0, \quad y = 0. \end{aligned}$$

It's easy to show the following limit statements are true:

$$\lim_{(x,y) \rightarrow (a,b)} x = a$$

$$\lim_{(x,y) \rightarrow (a,b)} y = b$$

$$\lim_{(x,y) \rightarrow (a,b)} c = c.$$

The limit laws for 1 variable also hold for 2 variables (or n variables).

Proposition: If $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L_1$, and $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = L_2$ then

$$1. \quad \lim_{(x,y) \rightarrow (a,b)} (f(x, y) \pm g(x, y)) = L_1 \pm L_2$$

$$2. \quad \lim_{(x,y) \rightarrow (a,b)} (f(x, y))(g(x, y)) = L_1 L_2$$

$$3. \quad \lim_{(x,y) \rightarrow (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L_1}{L_2} \text{ if } L_2 \neq 0$$

The squeeze theorem also holds:

Theorem: If $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L = \lim_{(x,y) \rightarrow (a,b)} g(x,y)$

and $f(x,y) \leq h(x,y) \leq g(x,y)$ then,

$$\lim_{(x,y) \rightarrow (a,b)} h(x,y) = L.$$

Ex. Use polar coordinates to find $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^2}$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r \rightarrow 0^+} \frac{(r \cos \theta)(r^2 \sin^2 \theta)}{r^2} = \lim_{r \rightarrow 0^+} r(\cos \theta)(\sin^2 \theta).$$

Since $-1 \leq \cos \theta \leq 1$ and $-1 \leq 0 \leq \sin^2 \theta \leq 1$:

$$-r \leq r(\cos \theta)(\sin^2 \theta) \leq r.$$

Now since $\lim_{r \rightarrow 0^+} (-r) = \lim_{r \rightarrow 0^+} (r) = 0$, by the squeeze theorem:

$$\lim_{r \rightarrow 0^+} r(\cos \theta)(\sin^2 \theta) = 0.$$

And thus $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^2} = 0$.

By limit laws all polynomials in two variables have limits everywhere (e.g. $f(x, y) = 3x^3 - 2x^2y + 6xy^3 - 2x + y - 3$) and are continuous.

Likewise, any rational function is continuous in its domain:

$$g(x, y) = \frac{2x^2 + 3xy - 4}{x^2 + y^2}.$$

Ex. Where is $f(x, y) = \frac{x^2y^2}{x^4 + y^4}$ continuous?

$f(x, y)$ has a domain of $\mathbb{R}^2 - (0, 0)$, so it's continuous there because it's a rational function.

Ex. Where is $f(x, y) = \frac{x^2y^2}{x^4 + y^4}$ $(x, y) \neq (0, 0)$
 $= 0$ $(x, y) = (0, 0)$

continuous?

$\mathbb{R}^2 - (0, 0)$ because we found $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2y^2}{x^4 + y^4}$ doesn't exist.

Ex. Where is $f(x, y) = \frac{4x^2y^2}{x^2+y^2}$ $(x, y) \neq (0,0)$
 $= 2$ $(x, y) = (0,0)$ continuous?

Is $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 2$?

Along $x = 0$, $f(0, y) = 0$ so, $f(0, y) \rightarrow 0$ as $(0, y) \rightarrow (0,0)$, so the limit can't be 2.

So $f(x, y)$ is continuous for $\mathbb{R}^2 - (0,0)$.

Notice that:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4x^2y^2}{x^2+y^2} = \lim_{r \rightarrow 0^+} \frac{4r^4 \cos^2 \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0^+} 4r^2 \cos^2 \theta \sin^2 \theta.$$

Since $0 \leq \cos^2 \theta \leq 1$ and $0 \leq \sin^2 \theta \leq 1$, we have:
 $0 \leq 4r^2 \cos^2 \theta \sin^2 \theta \leq 4r^2$.

Since $\lim_{r \rightarrow 0^+} (0) = \lim_{r \rightarrow 0^+} (4r^2) = 0$, by the squeeze theorem we have:
 $\lim_{r \rightarrow 0^+} 4r^2 \cos^2 \theta \sin^2 \theta = 0$.

So $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$.

Thus if $f(0,0) = 0$ then this function would have been continuous on all of $\mathbb{R}^2$.