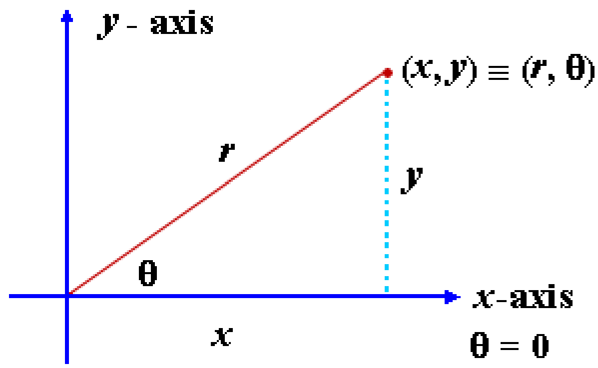
Cylindrical and Spherical Coordinates

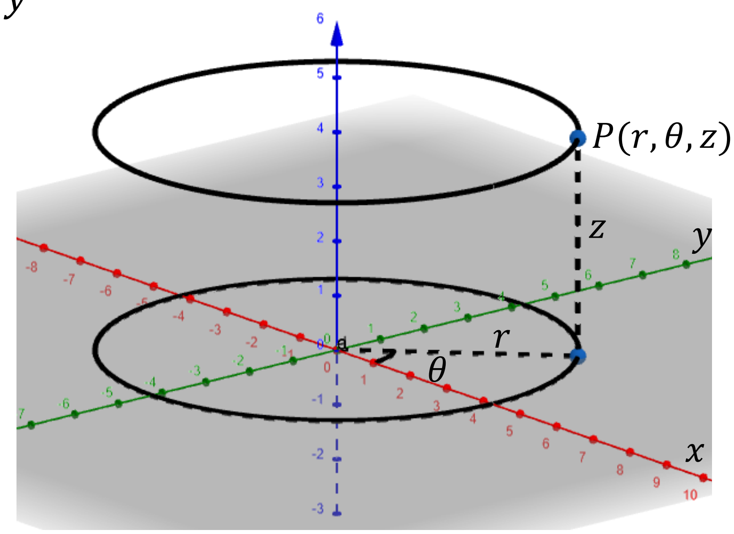
In 2 dimensions we have polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta \qquad 0 \le \theta < 2\pi$$
$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$



Cylindrical coordinates are just polar coordinates adding a z coordinate.

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$
$$z = z$$



Ex. If $\left(4, \frac{\pi}{6}, 3\right) = (r, \theta, z)$ are cylindrical coordinates, find the rectangular coordinates of the same point.

$$4 = r, \quad \frac{\pi}{6} = \theta, \quad 3 = z\,.$$

$$x = r\cos\theta = 4\cos\frac{\pi}{6} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$
$$y = r\sin\theta = 4\sin\frac{\pi}{6} = 4\left(\frac{1}{2}\right) = 2$$

Rectangular coordinates: $(x, y, z) = (2\sqrt{3}, 2, 3)$.

Ex. Find the cylindrical coordinates (r,θ,z) if the rectangular coordinates are $(x,y,z) = (2,2,5)$.

$\tan\theta = \frac{y}{x} = \frac{2}{2} = 1 \implies \theta = \frac{\pi}{4}, \frac{5\pi}{4}$.

$(2,2)$ is in the first quadrant so $\theta = \frac{\pi}{4}$.

$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

cylindrical coordinates: $(r,\theta,z) = (2\sqrt{2}, \frac{\pi}{4}, 5)$.

Cylindrical coordinates are useful when you have symmetry about an axis.

Ex. In cylindrical coordinates $r = 3$ is cylinder of radius 3, about the z axis.

Ex. $z^2 = x^2 + y^2$ is a cone, and in cylindrical coordinates it's written as $z^2 = r^2 \Rightarrow z = r$ (r can be $+$ or $-$).

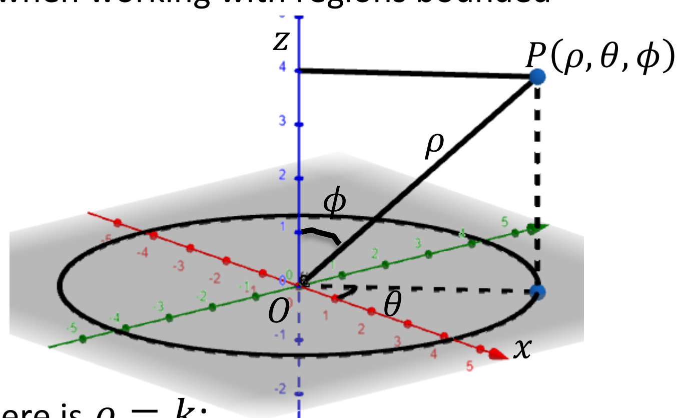
Spherical coordinates, $P(\rho,\theta,\phi)$, are useful when working with regions bounded by spheres or cones.

$\rho = |\overrightarrow{OP}|$

$\rho \geq 0$

$0 \leq \phi \leq \pi$

$0 \leq \theta < 2\pi$



In spherical coordinates the equation of a sphere is $\rho = k$:

$\theta = c$ is the equation of a vertical half plane.

$\phi = c$ is the equation of a half cone about the positive z axis if $0 < c < \frac{\pi}{2}$ and the negative z axis if $\frac{\pi}{2} < c < \pi$.

$\phi = \frac{\pi}{2}$ is the xy plane.

Relationship between rectangular and spherical coordinates:

$$P(x,y,z) = P(\rho,\theta,\phi)$$
$$\frac{z}{\rho} = \cos\phi \quad \Rightarrow \quad z = \rho\cos\phi$$
$$\frac{r}{\rho} = \sin\phi \quad \Rightarrow \quad r = \rho\sin\phi$$

$$\boxed{\begin{aligned} &x = r\cos\theta = \rho\sin\phi\cos\theta \\ &y = r\sin\theta = \rho\sin\phi\sin\theta \\ &z = \rho\cos\phi \\ &\rho^2 = x^2 + y^2 + z^2 \end{aligned}}$$

Ex. Convert the point $(\rho,\theta,\phi) = (3, \frac{\pi}{4}, \frac{\pi}{6})$ in spherical coordinates to rectangular coordinates, (x, y, z).

$$\rho = 3, \theta = \frac{\pi}{4}, \phi = \frac{\pi}{6}$$

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$x = 3\left(\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{4}\right) = 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{4}$$

$$y = 3\left(\sin\frac{\pi}{6}\right)\left(\sin\frac{\pi}{4}\right) = 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{4}$$

$$z = 3\cos\frac{\pi}{6} = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$$

$$(x,y,z) = \left(\frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{3}}{2}\right).$$

Ex. Convert the point $(x, y, z) = \left(1, \sqrt{3}, 2\sqrt{3}\right)$ in rectangular coordinates to a point in spherical coordinates, (ρ, θ, ϕ).

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 3 + 12} = \sqrt{16} = 4$$

$$z = \rho \cos\phi \;\Rightarrow 2\sqrt{3} = 4\cos\phi \;\Rightarrow \frac{\sqrt{3}}{2} = \cos\phi \;\Rightarrow\; \phi = \frac{\pi}{6}$$

$$x = \rho \sin\phi \cos\theta \;\Rightarrow\; 1 = 4\left(\sin\frac{\pi}{6}\right)\cos\theta$$

$$\Rightarrow 1 = 4\left(\frac{1}{2}\right)\cos\theta \;\Rightarrow \frac{1}{2} = \cos\theta \;\Rightarrow \theta = \frac{\pi}{3}\,,\frac{5\pi}{3}$$

$\left(1, \sqrt{3}\right)$ is in the first quadrant, so $\theta = \frac{\pi}{3}$ not $\frac{5\pi}{3}$.

So, $\quad (\rho, \theta, \phi) = \left(4, \frac{\pi}{3}\,, \frac{\pi}{6}\right)$.

Ex. Convert the point $(x, y, z) = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\sqrt{3})$ in rectangular coordinates to a point in spherical coordinates, (ρ, θ, ϕ).

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$$z = \rho \cos\phi \;\Rightarrow -\sqrt{3} = 2\cos\phi \;\Rightarrow -\frac{\sqrt{3}}{2} = \cos\phi \;\Rightarrow\; \phi = \frac{5\pi}{6}$$

$$x = \rho \sin\phi \cos\theta \;\Rightarrow\; -\frac{1}{2} = 2\left(\sin\frac{5\pi}{6}\right)\cos\theta$$

$$\Rightarrow -\frac{1}{2} = \cos\theta \;\Rightarrow -\frac{1}{2} = \cos\theta \;\Rightarrow \theta = \frac{2\pi}{3}\,,\frac{4\pi}{3}$$

$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ is in the second quadrant, so $\theta = \frac{2\pi}{3}$ not $\frac{4\pi}{3}$.

So, $\quad (\rho, \theta, \phi) = \left(2, \frac{2\pi}{3}\,, \frac{5\pi}{6}\right)$.