The Cross Product of Vectors in $\mathbb{R}^3$

The cross product is only defined for vectors in $\mathbb{R}^3$ and unlike the dot product, the answer is a vector in $\mathbb{R}^3$.

Def. If $\vec{A} = < a_1, a_2, a_3 >$, $\vec{B} = < b_1, b_2, b_3 >$, then:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

$$= (a_2b_3 - a_3b_2)\vec{\imath} + (a_3b_1 - a_1b_3)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}.$$

Recall:

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Ex. $$\begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 1(1-0) + 2(2-0) + 3(4-0)$$

$$= 1 + 4 + 12 = 17.$$

We can rewrite $\vec{A} \times \vec{B}$ as:

$$(a_2b_3 - a_3b_2)\vec{\imath} + (a_3b_1 - a_1b_3)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$\boldsymbol{\vec{A} \times \vec{B}} = \begin{vmatrix} \boldsymbol{\vec{\imath}} & \boldsymbol{\vec{\jmath}} & \boldsymbol{\vec{k}} \\ \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix}.$$

Ex. If $\vec{A} = < 2, 1, -2 >$, $\vec{B} = < -3, 2, 1 >$ find $\vec{A} \times \vec{B}$.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & -2 \\ -3 & 2 & 1 \end{vmatrix}$$

$$= \vec{\imath} \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix}$$

$$= \vec{\imath}(1 - (-4)) - \vec{\jmath}(2 - 6) + \vec{k}(4 - (-3))$$

$$= 5\vec{\imath} + 4\vec{\jmath} + 7\vec{k}$$

Ex. Show $\vec{A} \times \vec{A} = 0$

$$\vec{A} \times \vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \vec{\imath} \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix}$$

$$= (0)\vec{\imath} - (0)\vec{\jmath} + (0)\vec{k} = \vec{0}.$$

The most important property of $\vec{A} \times \vec{B}$ is the following:

Theorem: $\vec{A} \times \vec{B}$ is $\perp$ to $\vec{A}$ and $\vec{B}$.

Let's show $\vec{A} \times \vec{B} \perp$ to $\vec{A}$:

$$(\vec{A} \times \vec{B}) \cdot \vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot < a_1, a_2, a_3 >$$

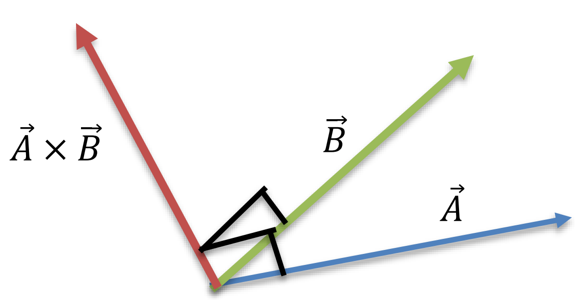
$$= (\vec{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}) \cdot < a_1, a_2, a_3 >$$

$$= (a_2b_3 - a_3b_2)a_1 - (a_1b_3 - a_3b_1)a_2 + (a_1b_2 - a_2b_1)a_3$$

$$= a_2b_3a_1 - a_3b_2a_1 + a_3b_1a_2 - a_1b_3a_2 + a_1b_2a_3 - a_2b_1a_3$$

$$= 0.$$

So if we think of vectors $\vec{A}$ and $\vec{B}$ with the same initial point, then $\vec{A} \times \vec{B}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. The direction of $\vec{A} \times \vec{B}$ is given by the right hand rule: if your fingers in your right hand curl from $\vec{A}$ to $\vec{B}$, then your thumb points in the direction of $\vec{A} \times \vec{B}$.



Theorem: If θ is the angle between $\vec{A}$ and $\vec{B}$ (so that $0 \le \theta \le \pi$), then

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta.$$

Proof:

$$\vec{A} \times \vec{B} = < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

$$\left|\vec{A} \times \vec{B}\right|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= \left|\vec{A}\right|^2\left|\vec{B}\right|^2 - \left(\vec{A} \cdot \vec{B}\right)^2$$

$$= \left|\vec{A}\right|^2\left|\vec{B}\right|^2 - \left(\left|\vec{A}\right| \left|\vec{B}\right| \cos\theta\right)^2$$

$$= \left|\vec{A}\right|^2\left|\vec{B}\right|^2 - \left|\vec{A}\right|^2 \left|\vec{B}\right|^2 \cos^2\theta$$

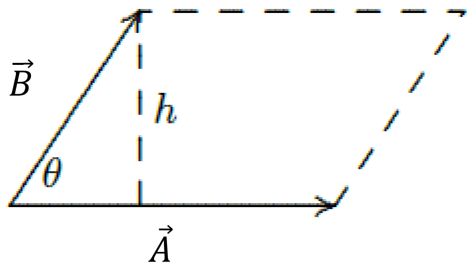
$$\left|\vec{A} \times \vec{B}\right|^2 = \left|\vec{A}\right|^2\left|\vec{B}\right|^2(1 - \cos^2\theta) = \left|\vec{A}\right|^2\left|\vec{B}\right|^2(\sin^2\theta)$$

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta.$$

Corollary: Two non-zero vectors are parallel $\Leftrightarrow \vec{A} \times \vec{B} = \vec{0}$.

Proof: $\vec{A}, \vec{B}$ parallel $\Leftrightarrow \theta = 0$ or $\theta = \pi$ (i.e. $\sin\theta = 0$).

Notice: Area of parallelogram $= |\vec{A}|(|\vec{B}|sin\theta) = |\vec{A} \times \vec{B}|$

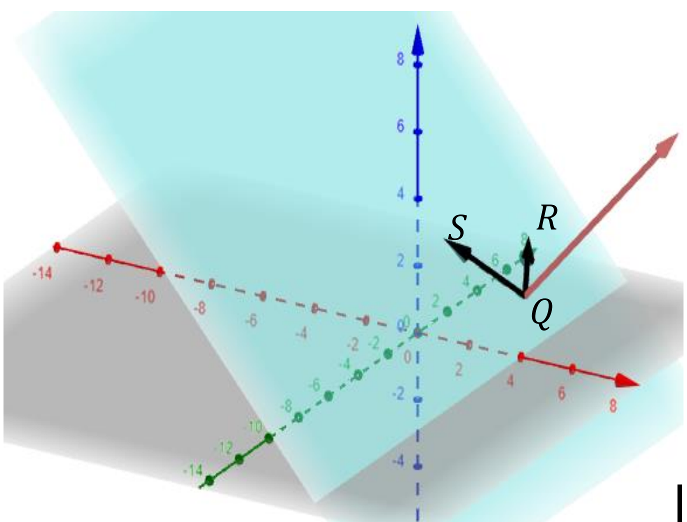


$h = |\vec{B}|sin\theta$

So the length of $\vec{A} \times \vec{B}$ is equal to the area of the parallelogram created by the vectors $\vec{A}$ and $\vec{B}$.

Ex. Find a vector perpendicular to the plane that passes through the points:

$$Q(3,2,1), \quad R(2,4,2), \quad S(1,0,3).$$



$$\overrightarrow{QS} = < -2, -2, 2 > = -2\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$$

$$\overrightarrow{QR} = < -1, 2, 1 > = -\vec{\imath} + 2\vec{\jmath} + \vec{k}$$

$$\overrightarrow{QR} \times \overrightarrow{QS} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 2 & 1 \\ -2 & -2 & 2 \end{vmatrix}$$

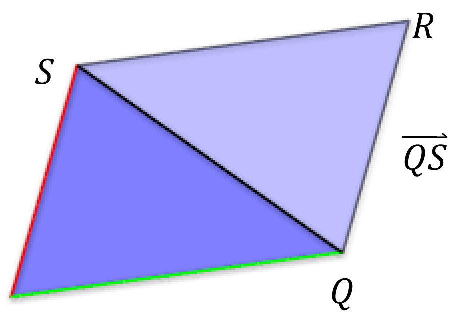
$$= \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} \vec{\imath} - \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} -1 & 2 \\ -2 & -2 \end{vmatrix} \vec{k}$$

$$= 6\vec{\imath} - (-2 - (-2))\vec{\jmath} + (2 + 4)\vec{k}.$$

$$\overrightarrow{QS} \times \overrightarrow{QR} = 6\vec{\imath} + 6\vec{k}.$$

Ex. Find the area of the triangle with vertices: $Q(3,2,1)$, $R(2,4,2)$, $S(1,0,3)$.

$\left|\overrightarrow{QS} \times \overrightarrow{QR}\right|$ = area of parallelogram= 2 times area of the triangle QRS



$\overrightarrow{QS} \times \overrightarrow{QR} = 6\vec{\imath} + 6\vec{k}$ (From previous example)

$$\left|\overrightarrow{QS} \times \overrightarrow{QR}\right| = \sqrt{(6)^2 + (0)^2 + (6)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$$

Area of the triangle $= \frac{1}{2}\, 6\sqrt{2} = 3\sqrt{2}.$

Notice: $\vec{\imath} \times \vec{\jmath} = \vec{k}$, $\quad \vec{\jmath} \times \vec{k} = \vec{\imath}$, $\quad \vec{k} \times \vec{\imath} = \vec{\jmath}$

$\vec{\jmath} \times \vec{\imath} = -\vec{k}$, $\quad \vec{k} \times \vec{\jmath} = -\vec{\imath}$, $\quad \vec{\imath} \times \vec{k} = -\vec{\jmath}$

So the cross product is not commutative. It's also not associative:

$$\vec{\imath} \times (\vec{\imath} \times \vec{\jmath}) = \vec{\imath} \times \vec{k} = -\vec{\jmath}$$

$$(\vec{\imath} \times \vec{\imath}\,) \times \vec{\jmath} = \vec{0} \times \vec{\jmath} = \vec{0}\,.$$

Theorem: $\vec{A}, \vec{B}, \vec{D}$ vectors, c is scalar

1. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
2. $(c\vec{A}) \times \vec{B} = c(\vec{A} \times \vec{B}) = \vec{A} \times c\vec{B}$
3. $\vec{A} \times (\vec{B} + \vec{D}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{D}$
4. $(\vec{A} + \vec{B}) \times \vec{D} = \vec{A} \times \vec{D} + \vec{B} \times \vec{D}$
5. $\vec{A} \cdot (\vec{B} \times \vec{D}) = (\vec{A} \times \vec{B}) \cdot \vec{D}$
6. $\vec{A} \times (\vec{B} \times \vec{D}) = (\vec{A} \cdot \vec{D})\vec{B} - (\vec{A} \cdot \vec{B})\vec{D}$.

Proof of #5: Scalar Triple Product

$$\vec{A} = < a_1, a_2, a_3 >, \vec{B} = < b_1, b_2, b_3 >, \vec{D} = < d_1, d_2, d_3 >$$

$$\vec{A} \cdot (\vec{B} \times \vec{D}) = < a_1, a_2, a_3 > \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

$$= < a_1, a_2, a_3 > \cdot \left(\begin{vmatrix} b_2 & b_3 \\ d_2 & d_3 \end{vmatrix} \vec{i} - \begin{vmatrix} b_1 & b_3 \\ d_1 & d_3 \end{vmatrix} \vec{j} + \begin{vmatrix} b_1 & b_2 \\ d_1 & d_2 \end{vmatrix} \vec{k} \right)$$

$$= < a_1, a_2, a_3 > \cdot < b_2 d_3 - d_2 b_3, -b_1 d_3 + d_1 b_3, b_1 d_2 - d_1 b_2 >$$

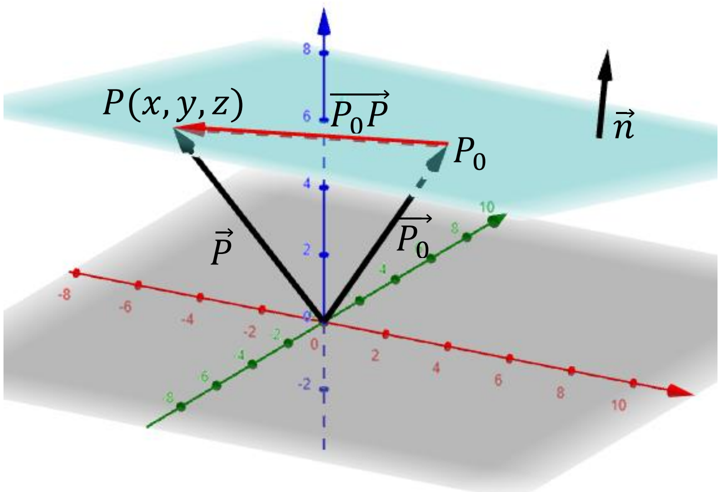
$$= a_1 b_2 d_3 - a_1 d_2 b_3 - a_2 b_1 d_3 + a_2 d_1 b_3 + a_3 b_1 d_2 - a_3 d_1 b_2$$

$$= d_1(a_2 b_3 - a_3 b_2) - d_2(a_1 b_3 - a_3 b_1) + d_3(a_1 b_2 - a_2 b_1)$$

$$= (\vec{A} \times \vec{B}) \cdot \vec{D}.$$

Planes in $\mathbb{R}^3$

A line is determined by a point and a direction vector. A plane in $\mathbb{R}^3$ is determined by a point, P_0, and a normal (⊥) vector, $\vec{n}$, to the plane. For any point, $P(x, y, z)$, in the plane the vector $\overrightarrow{P_0P}$ is in the plane and perpendicular to the normal vector, $\vec{n}$.



Vector equation of a plane:

$$\vec{n} \cdot (\vec{P} - \vec{P}_0) = 0$$

To get a scalar version, let:

$$\vec{n} = < a, b, c >, \quad \vec{P} = < x, y, z >, \quad \vec{P}_0 = < x_0, y_0, z_0 >$$

$$\vec{n} \cdot (\vec{P} - \vec{P}_0) = < a, b, c > \cdot < x - x_0, \; y - y_0, \; z - z_0 > = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

or

$$\boxed{ax + by + cz + d = 0}$$

This is the equation of a plane through (x_0, y_0, z_0) with normal vector $< a, b, c >$.

Note: if we know 3 points in a plane we can find 2 vectors in the plane and cross them to find a normal vector.

Ex. Find an equation of a plane through the point $(1, -1, 2)$ with a normal vector $< 3, 1, -4 >$.

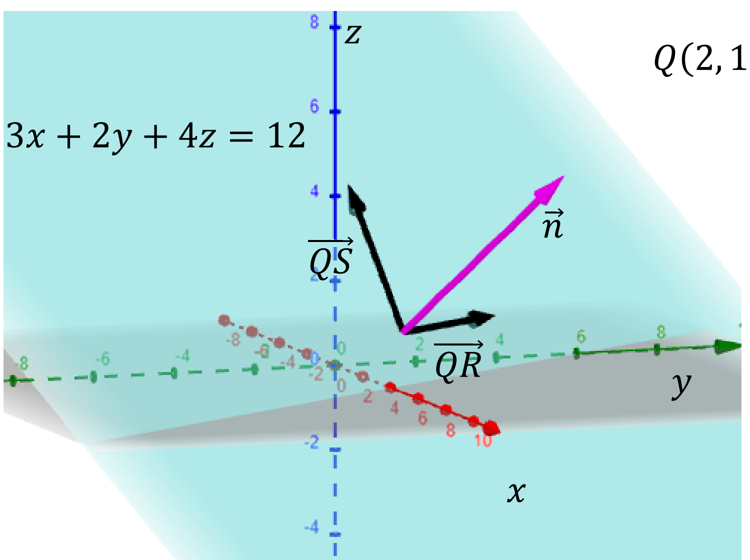
Equation of plane:

$$3(x-1)+1(y+1)-4(z-2)=0$$

$$3x-3+y+1-4z+8=0$$

$$3x+y-4z=-6\,.$$

Ex. Find the equation of the plane that passes through:



$Q(2,1,1),\ \ R(0,4,1),\ \ S(-2,1,4).$

$$\overrightarrow{QR} = < -2, 3, 0 >$$

$$\overrightarrow{QS} = < -4, 0, 3 >$$

$\overrightarrow{QR} \times \overrightarrow{QS}$ is normal to the plane containing $\overrightarrow{QR}$, $\overrightarrow{QS}$.

$$\vec{n} = \overrightarrow{QR} \times \overrightarrow{QS} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} \vec{\imath} - \begin{vmatrix} -2 & 0 \\ -4 & 3 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} -2 & 3 \\ -4 & 0 \end{vmatrix} \vec{k}$$

$$\vec{n} = 9\vec{\imath} + 6\vec{\jmath} + 12\vec{k}.$$

Using $(2, 1, 1)$ and $\vec{n} = 9\vec{\imath} + 6\vec{\jmath} + 12\vec{k}$:

$$9(x-2) + 6(y-1) + 12(z-1) = 0$$

$$9x - 18 + 6y - 6 + 12z - 12 = 0$$

$$9x + 6y + 12z - 36 = 0$$

Or $$3x + 2y + 4z = 12.$$

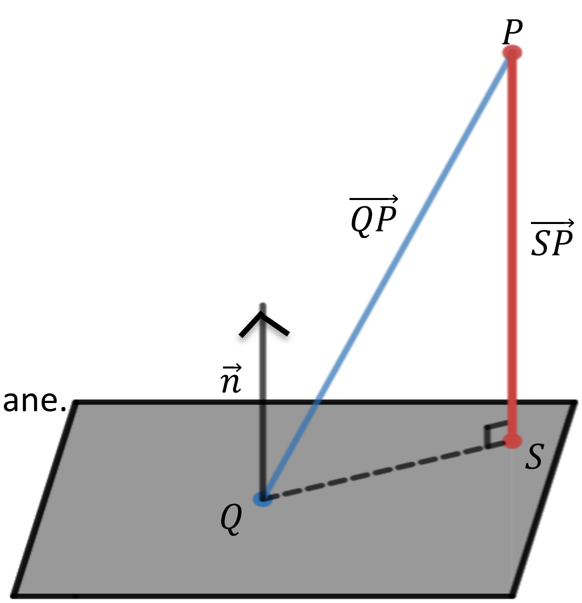
Note: 2 planes in $\mathbb{R}^3$ are parallel if their normal vectors are parallel.

Ex. Find the formula for the distance, D, from a point, $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Let $Q(x_0, y_0, z_0)$ be any point in the plane and let $\overrightarrow{QP}$ be the vector from Q to P.

$\overrightarrow{QP} = < x_1 - x_0, y_1 - y_0, z_1 - z_0 >$

$\vec{n} = < a, b, c >$ =normal vector to plane.



$D =$ absolute value of the scalar projection of $\overrightarrow{QP}$ onto $\vec{n}$, the normal vector to the plane.

$$D = \left|comp_{\vec{n}}\overrightarrow{QP}\right| = \left|\frac{\vec{n}\cdot\overrightarrow{QP}}{|\vec{n}|}\right|$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Since (x_0, y_0, z_0) is in the plane, it satisfies the equation:
$ax + by + cz + d = 0$, so

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Ex. Find the distance of the point $P(1, -2, 3)$ to the plane given by:

$$3x - y + 2z - 6 = 0.$$

$P = (1, -2, 3) = (x_1, y_1, z_1)$
$\vec{n} =< 3, -1, 2 >=< a, b, c >; \quad d = -6.$

$$D = \frac{|3(1) - 1(-2) + 2(3) - 6|}{\sqrt{3^2 + (-1)^2 + 2^2}} = \frac{|5|}{\sqrt{9 + 1 + 4}} = \frac{5}{\sqrt{14}}.$$

Ex. Find the distance of the point $P(-2,2,-1)$ to the plane given by:

$$2x + 4y - z = 7.$$

$$P = (-2, 2, -1) = (x_1, y_1, z_1)$$
$$\vec{n} =< 2, 4, -1 >=< a, b, c >; \quad d = -7.$$

$$D = \frac{|2(-2)+4(2)-1(-1)-7|}{\sqrt{2^2+(4)^2+(-1)^2}} = \frac{|-2|}{\sqrt{4+16+1}} = \frac{2}{\sqrt{21}}.$$