Arc Length

Given a curve in $\mathbb{R}^n$, $c(t)$, we want to find its length when $t_0 \leq t \leq t_1$. We define arc length as:

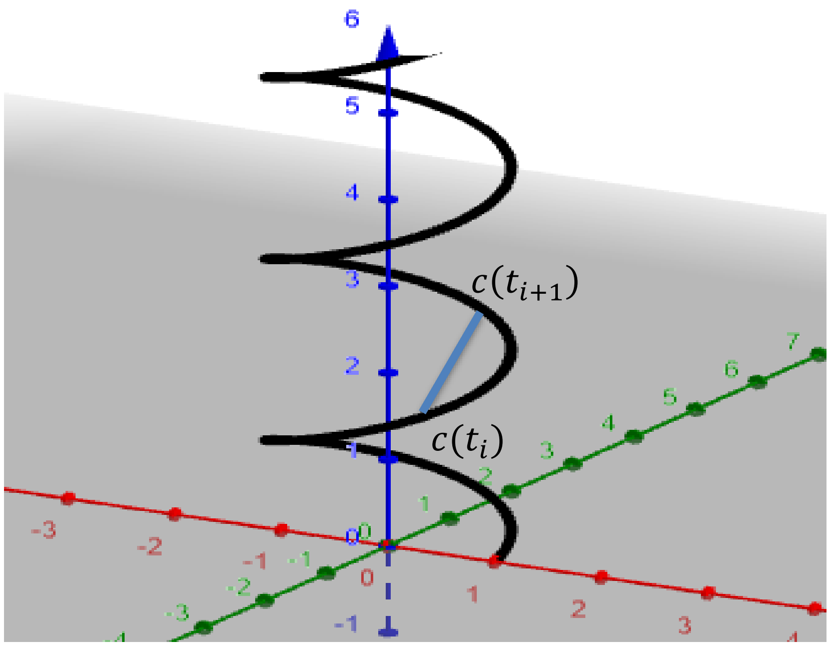
$$L(c) = \int_{t_0}^{t_1} \|c'(t)\| dt.$$

That is, we integrate the speed between $t = t_0$ and $t = t_1$ (this corresponds to our notion of arc length when the speed of an object is constant). We will motivate this definition in $\mathbb{R}^3$.

Let $c(t) = < x(t), y(t), z(t) >$. Now, let's take $a \leq t \leq b$ and subdivide the interval $[a, b]$ into N equal pieces:

$$a = t_0 < t_1 < \cdots < t_N = b$$

$$\Delta t = t_{i+1} - t_i = \frac{b-a}{N} \text{ for } 0 \leq i \leq N - 1.$$



Notice for any interval $t_i \leq t \leq t_{i+1}$:

$$\|c(t_{i+1}) - c(t_i)\|$$
$$= \sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2 + [z(t_{i+1}) - z(t_i)]^2}.$$

By the mean value theorem there are points in $[t_i, t_{i+1}]$, t_i^*, t_i^{**}, and t_i^{***} with:

$$x(t_{i+1}) - x(t_i) = x'(t_i^*)(t_{i+1} - t_i)$$

$$y(t_{i+1}) - y(t_i) = y'(t_i^{**})(t_{i+1} - t_i)$$

$$z(t_{i+1}) - z(t_i) = z'(t_i^{***})(t_{i+1} - t_i).$$

Thus:

$$\|c(t_{i+1}) - c(t_i)\| = \sqrt{\left(x'(t_i^*)\right)^2 + \left(y'(t_i^{**})\right)^2 + \left(z'(t_i^{***})\right)^2}\ (t_{i+1} - t_i).$$

So the approximate length of the curve is:

$$S_N \approx \sum_{i=0}^{N-1} \sqrt{\left(x'(t_i^*)\right)^2 + \left(y'(t_i^{**})\right)^2 + \left(z'(t_i^{***})\right)^2}\ (t_{i+1} - t_i)$$

$$L(c) = \lim_{N\to\infty} S_N = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\ dt$$

Ex. Find the length of the curve $c(t) = < \sin(3t) , \cos(3t) , 2t^{\frac{3}{2}} >$ for $0 \leq t \leq 1$.

$$L(c) = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$c'(t) = < 3\cos(3t) , -3\sin(3t) , 3t^{\frac{1}{2}} >$$

$$\frac{dx}{dt} = 3\cos(3t)$$

$$\frac{dy}{dt} = -3\sin(3t)$$

$$\frac{dz}{dt} = 3t^{\frac{1}{2}}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 9t}$$

$$= \sqrt{9 + 9t}$$

$$L(c) = \int_0^1 \sqrt{9 + 9t}\, dt = \int_0^1 (9 + 9t)^{\frac{1}{2}}\, dt$$

Let:

$u = 9 + 9t$ when $t = 0,\ u = 9$; when $t = 1,\ u = 18.$

$du = 9\, dt$

$\frac{1}{9}\, du = dt$

$$L(c) = \int_{u=9}^{u=18} (u)^{\frac{1}{2}} \left(\frac{1}{9}\right) du = \frac{1}{9}\left(\frac{2}{3}\right) u^{\frac{3}{2}}\Big|_{u=9}^{u=18}$$

$$= \frac{2}{27}\left[(18)^{\frac{3}{2}} - (9)^{\frac{3}{2}}\right]$$

$$= \frac{2}{27}\left[(2)^{\frac{3}{2}} \cdot (9)^{\frac{3}{2}} - (9)^{\frac{3}{2}}\right]$$

$$= \frac{2}{27}\left[27\left(2\sqrt{2}\right) - 27\right]$$

$$= 4\sqrt{2} - 2.$$

Ex. Let $c(t) = < 2t, t^2, \ln t >$ be defined for $t > 0$. Find the arc length of c between the points $(2, 1, 0)$ and $(4, 4, \ln 2)$.

At $(2, 1, 0)$: $2t = 2,\ \ t^2 = 1,$ and $\ln t = 0 \quad \Rightarrow \quad t = 1.$

At $(4, 4, \ln 2)$: $2t = 4,\ \ t^2 = 4,$ and $\ln t = \ln 2 \quad \Rightarrow \quad t = 2.$

$$L(c) = \int_1^2 \|c'(t)\| dt$$

$$c'(t) = < 2, 2t, \frac{1}{t} >$$

$$\|c'(t)\| = \sqrt{4 + 4t^2 + \frac{1}{t^2}} = \sqrt{\frac{4t^2+4t^4+1}{t^2}}$$

$$= \frac{\sqrt{4t^4+4t^2+1}}{\sqrt{t^2}} = \frac{\sqrt{(2t^2+1)^2}}{\sqrt{t^2}} = \frac{2t^2+1}{t}$$

$$= 2t + \frac{1}{t}.$$

$$L(c) = \int_{t=1}^{t=2} \left(2t + \frac{1}{t}\right) dt = (t^2 + \ln(t))\Big|_{t=1}^{t=2}$$

$$= (4 + \ln(2)) - (1 + \ln(1))$$

$$= 3 + \ln(2).$$

Ex. Find the length of $c(t) =< 3t, 4t, 5\cosh(t) >$ where $0 \leq t \leq ln2.$

Recall that: $\cosh(t) = \frac{e^t+e^{-t}}{2}$, $\sinh(t) = \frac{e^t-e^{-t}}{2}$;

$\cosh^2(t) - \sinh^2(t) = 1$ or $\cosh^2(t) = 1 + \sinh^2(t)$;

$\frac{d}{dt}(\cosh(t)) = \sinh(t)$, $\frac{d}{dt}(\sinh(t)) = \cosh(t)$.

$c'(t) =< 3,\ 4, 5\sinh(t) >.$

$$\|c'(t)\| = \sqrt{3^2 + 4^2 + 25\sinh^2(t)} = \sqrt{25 + 25\sinh^2(t)}$$
$$= 5\sqrt{1 + \sinh^2(t)} = 5\cosh(t).$$

$$L(c) = \int_{t=0}^{t=ln2} 5\cosh(t)dt = 5\sinh(t)|_{t=0}^{t=ln2}$$

$$= 5(\frac{e^t - e^{-t}}{2})\Big|_{t=0}^{t=ln2}$$

$$= 5[\left(\frac{e^{ln2} - e^{-ln2}}{2}\right) - \left(\frac{e^0 - e^0}{2}\right)]$$

$$= 5\left[\frac{2 - \frac{1}{2}}{2} - 0\right] = \frac{15}{4}.$$