The Implicit Function Theorem and the Inverse Function Theorem

Not all relations, even in two variables, $f(x, y) = 0$, can be written $y = f(x)$. Sometimes functions are defined implicitly. For example: $2x^3 + xy^5 + y^3 = 0$ implicitly defines a function, y, in terms of x.

However, as we saw in first year calculus, even though we don't have an explicit formula for $y = f(x)$, we may still be able to find $\frac{dy}{dx}$ through implicit differentiation.

Example: Find $\frac{dy}{dx}$ for the curve given by $2x^3 + xy^5 + y^3 = 0$.

$$\frac{d}{dx}\left(2x^3 + xy^5 + y^3\right) = \frac{d}{dx}(0)$$

$$6x^2 + x(5y^4)\frac{dy}{dx} + y^5(1) + 3y^2\frac{dy}{dx} = 0$$

$$(5xy^4 + 3y^2)\frac{dy}{dx} + (6x^2 + y^5) = 0$$

$$(5xy^4 + 3y^2)\frac{dy}{dx} = -(6x^2 + y^5)$$

$$\frac{dy}{dx} = -\frac{6x^2+y^5}{5xy^4+3y^2}\,.$$

For example, the point $(1, -1)$ lies on the curve given by $2x^3 + xy^5 + y^3 = 0$ (because $(1, -1)$ satisfies this equation) and the slope of the tangent line to this curve at $(1, -1)$ is given by:

$$\frac{dy}{dx} = -\frac{6x^2+y^5}{5xy^4+3y^2} = -\frac{6-1}{5+3} = -\frac{5}{8}\,.$$

Notice that if we consider the function of 2 variables:

$$f(x, y) = 2x^3 + xy^5 + y^3$$

The derivative in our last example, $\frac{dy}{dx} = -\frac{6x^2+y^5}{5xy^4+3y^2}$, is just:

$$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}\right),$$

Since $\frac{\partial f}{\partial x} = 6x^2 + y^5$ and $\frac{\partial f}{\partial y} = 5xy^4 + 3y^2$.

Similarly, given a surface in $\mathbb{R}^3$ described implicitly by an equation $f(x, y, z) = 0$ we would like to be able to calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ even if we can't solve $f(x, y, z) = 0$ for $z = g(x, y)$. For example, suppose we have a surface in $\mathbb{R}^3$ given implicitly by the equation $x + y - z + \cos(xyz) = 0$, how can we find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$? We can do this with the Implicit Function Theorem:

Theorem (Implicit Function Theorem): Suppose $F: \mathbb{R}^3 \to \mathbb{R}$ where

$$F(x_0, y_0, z_0) = 0 \text{ and } \frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0.$$

Then there exists a region in $\mathbb{R}^2$, U, containing (x_0, y_0) and a neighborhood, V, of z_0 in $\mathbb{R}$, such that there is a unique function $z = g(x, y)$ defined on $(x, y) \in U$ and $z \in V$ such that $F\big(x, y, g(x, y)\big) = 0$. In addition, if $(x, y) \in U$ and $z \in V$ satisfies $F(x, y, z) = 0$ then $z = g(x, y)$ and

$$\frac{\partial z}{\partial x} = \frac{\partial g}{\partial x} = -\left(\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z}\right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial g}{\partial y} = -\left(\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z}\right).$$

Example: Suppose we have a surface in $\mathbb{R}^3$ given implicitly by the equation $x + y - z + \cos(xyz) = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at any point (x, y, z) on this surface. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x, y) = (0,0)$.

$$F(x, y, z) = x + y - z + \cos(xyz)$$

$$\frac{\partial F}{\partial x} = 1 - yzsin(xyz), \qquad \frac{\partial F}{\partial y} = 1 - xzsin(xyz),$$

$$\frac{\partial F}{\partial z} = -1 - xysin(xyz).$$

By the implicit function theorem we know:

$$\frac{\partial z}{\partial x} = \frac{\partial g}{\partial x} = -\left(\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z}\right) \qquad \frac{\partial z}{\partial y} = \frac{\partial g}{\partial y} = -\left(\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z}\right).$$

Plugging into these formulas we get:

$$\frac{\partial z}{\partial x} = -\left(\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z}\right) = -\frac{1 - yzsin(xyz)}{-1 - xysin(xyz)}$$

$$\frac{\partial z}{\partial y} = -\left(\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z}\right) = -\frac{1 - xzsin(xyz)}{-1 - xysin(xyz)} \ .$$

At $(0,0)$: $\frac{\partial z}{\partial x} = -\frac{1}{-1} = 1, \qquad \frac{\partial z}{\partial y} = -\frac{1}{-1} = 1.$

Notice that the implicit function theorem says that if we have a function, $F(x, y, z) = 0$, and $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$, then locally around$(x_0, y_0, z_0)$ the graph of $F(x, y, z) = 0$ looks like $z = g(x, y)$, where g has a differentiable inverse. Thus if $F(x, y, z) = 0$ has the property that $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$ for any point where $F(x, y, z) = 0$, then $F(x, y, z) = 0$ is a differentiable surface.

In first year calculus, we learn that if $f: \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $f'(a) \neq 0$, then there is an open interval, V, containing a such that $f'(x) > 0$ or $f'(x) < 0$ for all $x \in V$.
If $f'(x) > 0$, then f is strictly increasing on V. If $f'(x) < 0$, then f is strictly decreasing on V. Therefore, f is 1-1 on V and has an inverse function on $f(V) = W$. In addition, if $y \in W$ then,

$$(f^{-1})'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

Thus if we can find the derivative of $f(x)$ and we know the value of $f^{-1}(y)$, at a fixed point y, we can find $(f^{-1})'(y)$ without having a concrete formula for the function $f^{-1}(y)$ for any y.

Ex. Let $f(x) = 3x + \cos(x)$. Find $(f^{-1})'(1)$.

Since $f'(x) = 3 - \sin(x) > 0$, we know that $f(x)$ is strictly increasing and hence 1-1, so it has an inverse function.

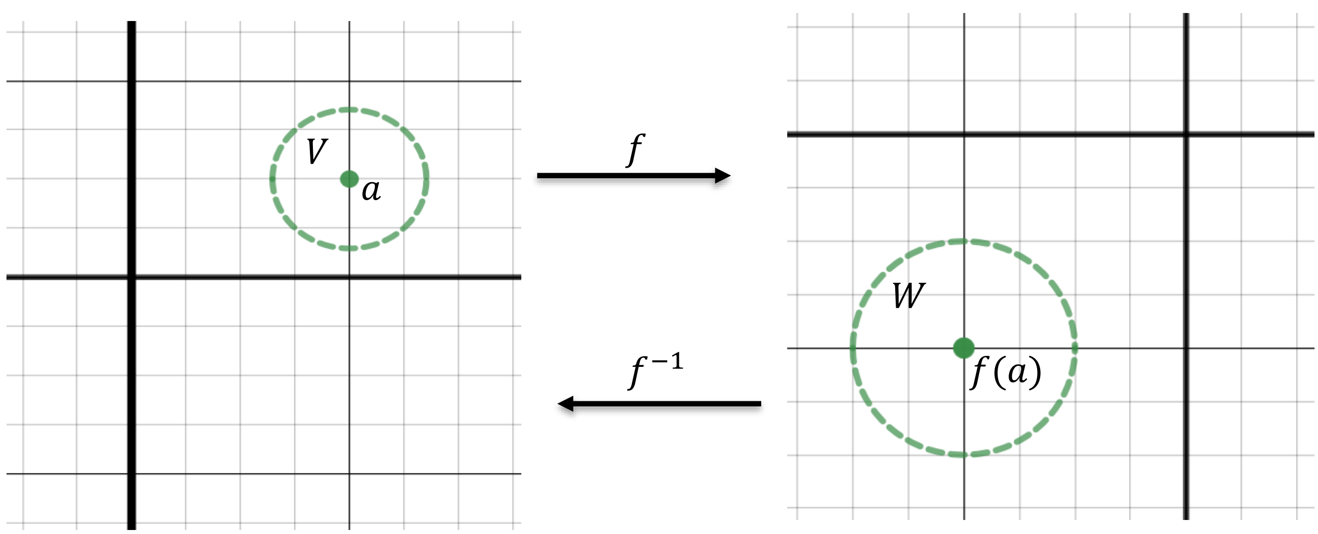
$f(0) = 1$, so $f^{-1}(1) = 0$.

$$(f^{-1})'(1) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{f'(0)} = \frac{1}{3-0} = \frac{1}{3}.$$

We would like to develop a similar theorem for $f: \mathbb{R}^n \to \mathbb{R}^n$.

Inverse Function Theorem: Suppose that $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing a and $\det\big(Df(a)\big) \neq 0$, then there is an open set, V, containing a and an open set, W, containing $f(a)$ such that $f: V \to W$ has a continuous inverse, $f^{-1}: W \to V$, which is differentiable for $y \in W$ and satisfies:

$$(f^{-1})'(y) = \left[f'\big(f^{-1}(y)\big)\right]^{-1}.$$



Ex. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ by $F(s,t) = (s^2 - t^2, 2st)$. Show that there exists an open set, V, containing $(2,3)$ and an open set, W, containing $F(2,3) = (-5,12)$ such that F has a continuously differentiable inverse $F^{-1}: W \to V$. Find $DF^{-1}(-5,12)$ and show F does not have an inverse globally.

$$DF(s,t) = \begin{pmatrix} 2s & -2t \\ 2t & 2s \end{pmatrix}$$

so $F(s,t)$ is continuously differentiable everywhere since all of the partial derivatives are continuous everywhere.

$$DF(2,3) = \begin{pmatrix} 4 & -6 \\ 6 & 4 \end{pmatrix}$$

$$\det\big(DF(2,3)\big) = 16 + 36 = 52 \neq 0$$

So by the inverse function theorem, there exist open sets, V and W, containing $(2,3)$ and $(-5,12)$ such that $F^{-1}: W \to V$ and F^{-1} is continuously differentiable.

$$DF^{-1}(-5,12) = [DF(2,3)]^{-1}$$

$$DF^{-1}(-5,12) = \frac{1}{52}\begin{pmatrix} 4 & 6 \\ -6 & 4 \end{pmatrix}$$

$$DF^{-1}(-5,12) = \frac{1}{26}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

For F to have a global inverse, it would need to be 1-1 on all of $\mathbb{R}^2$. But $F(-1,-1) = (0,2)$ and $F(1,1) = (0,2)$, so F is not globally 1-1 and hence has no global inverse.

The inverse function theorem only guarantees a local inverse. In fact, f can have a local inverse at every point and not have a global inverse.