The Metric Space $C(I)$

Def. Let $\boldsymbol{C(I)} = \{$**bounded continuous functions** $\boldsymbol{f}: \boldsymbol{I} \subseteq \mathbb{R} \to \mathbb{R}\}$

Note: If I is closed and bounded then any continuous function is bounded on I.

$C(I)$ is a metric space with the distance defined as:

$$d\big(f(x), g(x)\big) = \sup_{x \in I} |f(x) - g(x)|$$

1. $d\big(f(x), g(x)\big) = \sup_{x \in I} |f(x) - g(x)| \geq 0$; and
   $d\big(f(x), g(x)\big) = 0$ implies $f(x) = g(x)$.

2. $d\big(f(x), g(x)\big) = d\big(g(x), f(x)\big)$.

3. $d\big(f(x), g(x)\big) \leq d\big(f(x), h(x)\big) + d\big(h(x), g(x)\big)$.

   This is true because if $A(x) = B(x) + E(x)$ then by the triangle inequality:
   $|A(x)| \leq |B(x)| + |E(x)|$ for any $x \in I$.

   Thus we have: $\sup_{x \in I} |A(x)| \leq \sup_{x \in I} |B(x)| + \sup_{x \in I} |E(x)|$.

Now let $A(x) = f(x) - g(x)$, $B(x) = f(x) - h(x)$, $E(x) = h(x) - g(x)$.

This gives us: $d\big(f(x), g(x)\big) \leq d\big(f(x), h(x)\big) + d\big(h(x), g(x)\big)$.

Notice that a sequence of functions $f_n(x) \in C(I)$ converges to $f(x)$ with this metric if given any $\epsilon > 0$ there exists a $N \in \mathbb{Z}^+$ such that if $n \geq N$ then
$d\big(f_n(x), f(x)\big) = \sup_{x \in I} |f_n(x) - f(x)| < \epsilon$.

This ϵ statement is equivalent to saying that $|f_n(x) - f(x)| < \epsilon$ for all $x \in I$.
Thus convergence in $C(I)$ is the same as uniform convergence.

We already know that if $f_n(x)$ converges uniformly to $f(x)$ and all of the $f_n(x)$ are continuous then so is $f(x)$. Our goal is to show that $C(I)$ is complete with the metric described above. Thus we must show that every Cauchy sequence in $C(I)$ converges in $C(I)$. Later in this section we'll see that if $\{f_n(x)\}$ is a Cauchy sequence in $C(I)$ then $\{f_n(x)\}$ converges uniformly to a function $f(x)$. Since each f_n is continuous, f must also be continuous. We will then show that since each f_n is bounded on I and $\{f_n(x)\}$ and converges uniformly to a function $f(x)$, then $f(x)$ is also bounded on I. Thus any Cauchy sequence in $C(I)$ converges to a function in $C(I)$. Hence $C(I)$ is a complete metric space.

Recall that:

Def. Let V be a vector (linear) space. A real valued function $\|\cdot\|$ on V is called a **norm** if for each $v, w \in V$ and $\alpha \in \mathbb{R}$:

1. $\|v + w\| \leq \|v\| + \|w\|$ (Triangle inequality)
2. $\|\alpha v\| = |\alpha|\|v\|$ (positive homogenity)
3. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.

Ex. $\mathbb{R}^n$ is a normed linear space with $v = < a_1, \dots, a_n > \in \mathbb{R}^n$; and

$$\|v\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\ .$$

Given any normed vector space V we can always define a metric on V by

$$d(v, w) = \|v - w\|.$$

Ex. $C(I)$ is a vector space. We can define a norm on $C(I)$ by

$$\|f\|_\infty = \sup_{x \in I} |f(x)|\,, \quad f \in C(I).$$

Def. Given a sequence $\{v_n\} \subseteq V$, a normed vector space, we say that $\{\boldsymbol{v_n}\}$ **converges to $\boldsymbol{v}$ with respect to the norm** $\|\cdot\|$ on V if $\{v_n\}$ converges to v with respect to the metric $d(v,w) = \|v - w\|$. That is, given any $\epsilon > 0$ there exists a $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $d(v_n, v) = \|v_n - v\| < \epsilon$.

Def. Given a normed vector space V, we say that **$\boldsymbol{V}$ is complete with respect to** $\|\cdot\|$ if V is complete with respect to the metric $d(v,w) = \|v - w\|$.

Def. A complete normed vector space is called a **Banach space**.

Ex. $\mathbb{R}^n$ is a Banach space with the standard norm on $\mathbb{R}^n$.

Ex. $C(I)$ is a Banach space with $d\big(f(x), g(x)\big) = \sup_{x \in I} |f(x) - g(x)|$. We will see this shortly.

Def. Let V and W be normed vector spaces. A linear transformation, T, from V to W is called an **operator from $\boldsymbol{V}$ to $\boldsymbol{W}$.** T is called **bounded** if there is an $M \in \mathbb{R}$ such that :

$$\|T(v)\|_W \leq M\|v\|_V \quad \text{for all } v \in V.$$

Ex. Let $T: C[0,3] \to \mathbb{R}$ by $T(f) = \int_0^3 f(x)dx$.

T is bounded because:

$$\|T(f)\| = \left\|\int_0^3 f(x)dx\right\| \leq (3-0) \sup_{0 \leq x \leq 3} |f(x)| = 3\,\|f\|_\infty.$$

The set of bounded linear operators from V to W, $\mathcal{L}(V,W)$, is also a normed vector space. We can define a norm on $\mathcal{L}(V,W)$ by

$$\|T\| = \inf\{M \mid \|T(v)\|_W \le M\|v\|_V \text{ for all } v \in V\}.$$

This norm is often called the **operator norm**.

Ex. Let $T: C[0,3] \to \mathbb{R}$ by $T(f) = \int_0^3 f(x)dx$. Then

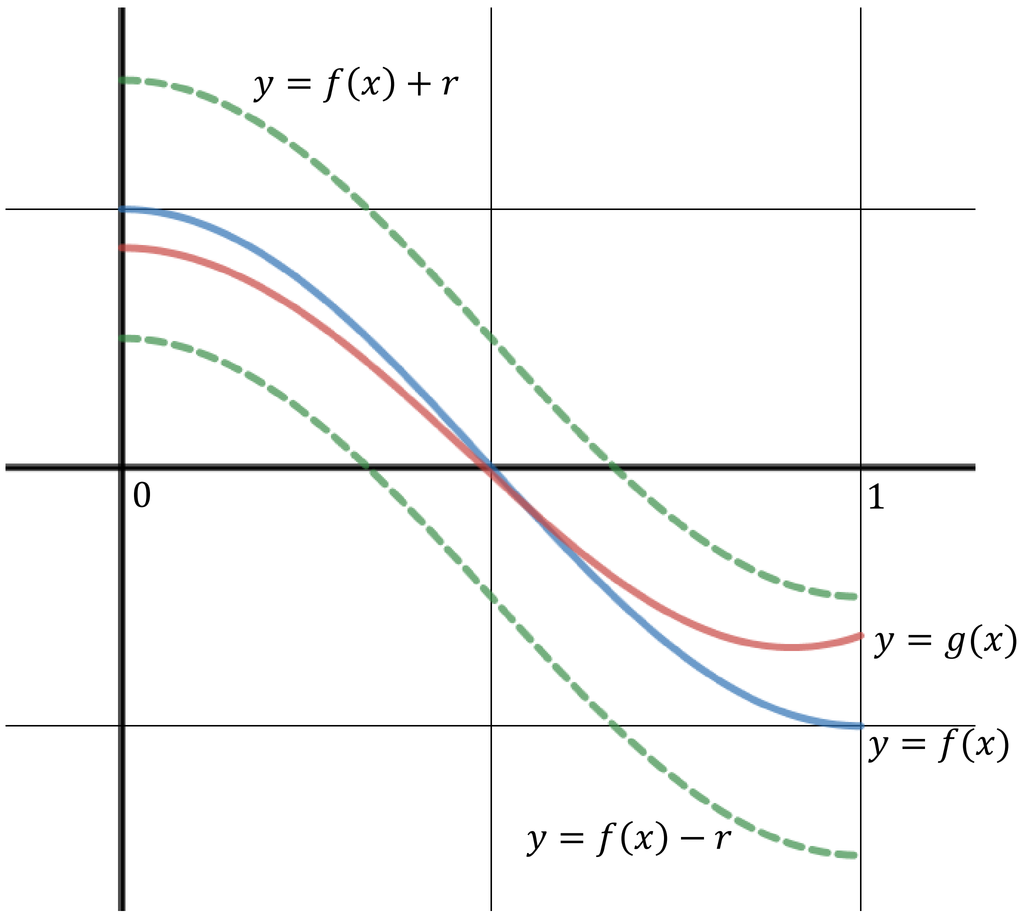
$$\|T\| = \inf\{M \mid \left\|\int_0^3 f(x)dx\right\| \le M\|f\|_\infty \text{ for all } f \in C[0,3]\}.$$

From the previous example we know that $\|T\| \le 3$. However, we also know that $f(x) = 1 \in C[0,3]$ and $T(f) = \int_0^3 1dx = 3 = 3\,\|f\|_\infty$, so $\|T\| = 3$.

Ex. Describe $N_r(f)$, a neighborhood of radius r centered at $f \in C[0,1]$.

$$N_r(f) = \{g \in C[0,1] \mid \sup_{0 \le x \le 1} |f(x) - g(x)| < r\}$$

$$= \{g \in C[0,1] \mid f(x) - r < g(x) < f(x) + r\}$$



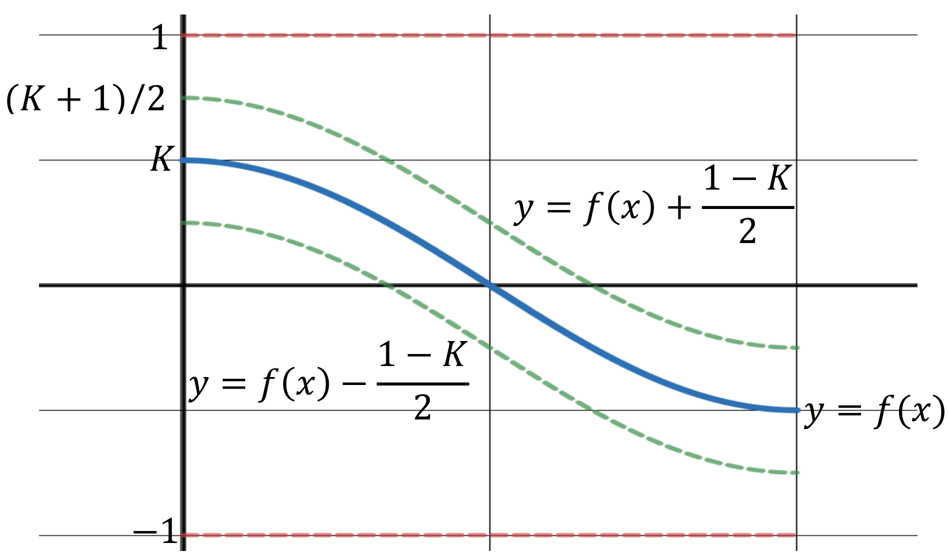
$y = g(x)$ is in $N_r(f)$ if the graph of $y = g(x)$ lies between the dotted green curves given by $y = f(x) + r$ and $y = f(x) + r$ when $0 \le x \le 1$.

Ex. Let $E \subseteq C[0,1]$, with $E = \{f \in C[0,1] | \sup_{0\le x\le 1} |f(x)| < 1\}$. Notice that $E = B_1(g(x) = 0)$, the ball of radius 1 around $g(x) = 0$ in $C[0,1]$.

a. Is E open in $C[0,1]$? If so, prove it.
b. If $g \in C[0,1]$ and $\sup_{0\le x\le 1} |g(x)| = 1$, is g a limit point of E?
c. Is E bounded in $C[0,1]$?
d. Is E totally bounded in $C[0,1]$?

a. Yes, E is open in $C[0,1]$. To prove this we must show that every element of E is an interior point. That is, given any $f \in E$, there exists a neighborhood of f, $N_r(f)$, such that $N_r(f) \subseteq E$.

$$N_r(f) = \{g \in C[0,1] | d(f,g) < r\}$$
$$= \{g \in C[0,1] | \sup_{0\le x\le 1} |f(x) - g(x)| < r\}$$



Since $|f(x)|$ is a continuous function on $[0,1]$ it attains its maximum value. Let $\sup_{0\le x\le 1} |f(x)| = K < 1$.

Choose $r = \frac{1-K}{2}$.

Then: $N_{\frac{1-K}{2}}(f) = \{g \in C[0,1] | \sup_{0\le x\le 1} |f(x) - g(x)| < \frac{1-K}{2}\}$.

Claim: $N_{\frac{1-K}{2}}(f) \subseteq E$.

We have to show that for any $g \in N_{\frac{1-K}{2}}(f)$, g is also in E. That is,

we need to show that $\sup_{0\le x\le 1} |g(x)| < 1$.

$$g \in N_{\frac{1-K}{2}}(f) \Rightarrow \sup_{0\le x\le 1} |g(x) - f(x)| < \frac{1-K}{2}$$

$$\frac{K-1}{2} < g(x) - f(x) < \frac{1-K}{2}; \quad \text{for all } 0 \le x \le 1.$$

$$f(x) + \frac{K-1}{2} < g(x) < f(x) + \frac{1-K}{2}$$

Since $\sup_{0\le x\le 1} |f(x)| = K, \quad -K \le f(x) \le K;$ for all $0 \le x \le 1$.

Thus we have:

$$-K + \frac{K-1}{2} \le f(x) + \frac{K-1}{2} < g(x) < f(x) + \frac{1-K}{2} \le K + \frac{1-K}{2}$$

$$-(\frac{K+1}{2}) \le f(x) + \frac{K-1}{2} < g(x) < f(x) + \frac{1-K}{2} \le \frac{K+1}{2}$$

But since $0 \le K < 1$ we have:

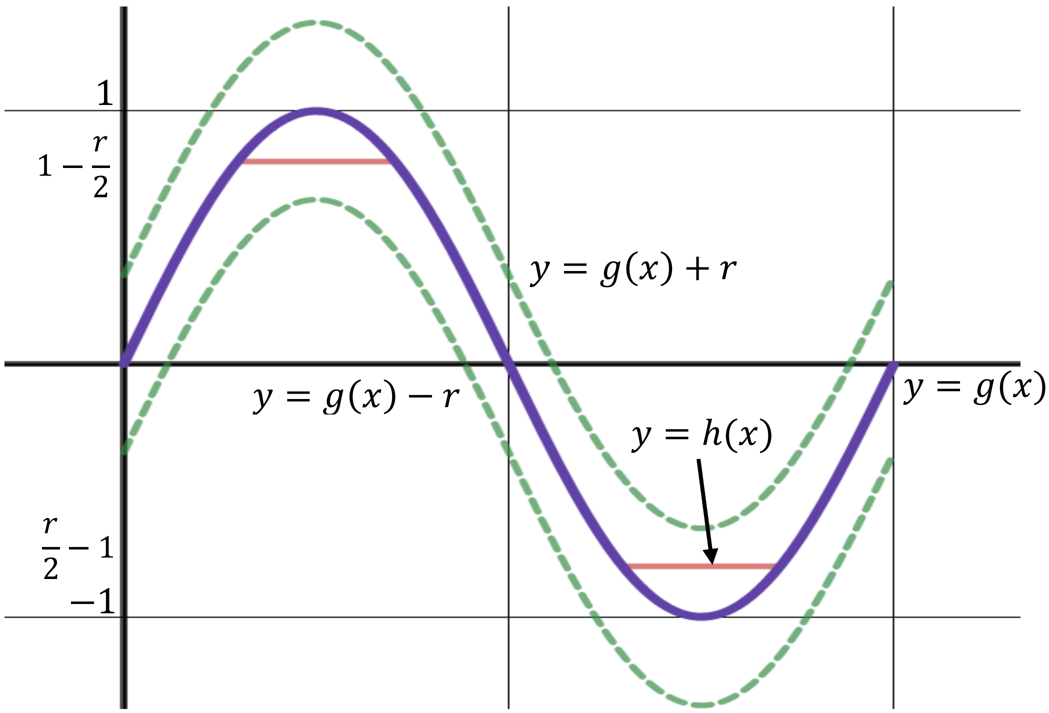
$$-1 < -\left(\frac{K+1}{2}\right) \le f(x) + \frac{K-1}{2} < g(x) < f(x) + \frac{1-K}{2} \le \frac{K+1}{2} < 1.$$

These inequalities hold for all $0 \le x \le 1$, so $\sup_{0\le x\le 1} |g(x)| < 1$, and $g(x) \in E$.

Thus $N_{\frac{1-K}{2}}(f) \subseteq E$ and E is an open set in $C[0,1]$.

b. If $\sup_{0\le x\le 1} |g(x)| = 1$, let's show $g(x)$ is a limit point of E.

To be a limit point we must show that every neighborhood of $g(x)$, $N_r(g)$, intersects E in some point other than g.



We can always construct a function $h \in C[0,1]$ such that

$$h(x) = \begin{cases} g(x) & \text{if } |g(x)| < 1 - \frac{r}{2} \\ 1 - \frac{r}{2} & \text{if } g(x) \ge 1 - \frac{r}{2} \\ \frac{r}{2} - 1 & \text{if } g(x) \le \frac{r}{2} - 1. \end{cases}$$

Now if we take $r < \frac{1}{2}$, then $h(x) \in E$ and $N_r(g)$.

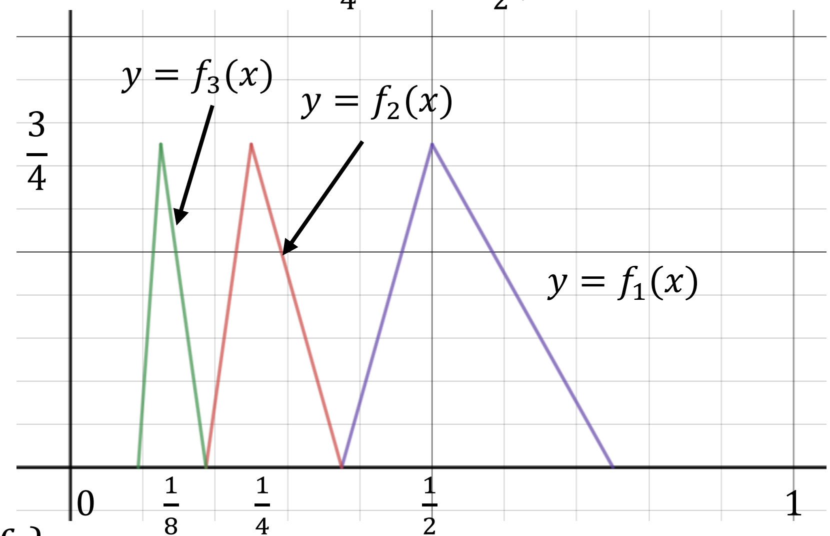
If $r \ge \frac{1}{2}$, take $h(x)$ defined with $r = \frac{1}{2}$, then $h(x) \in E$ and $N_r(g)$.

So $g(x)$ is a limit point of E.

c. Yes, E is bounded because for any $f \in E$, $d(f, g(x) = 0) < 1$.

d. No, E is not totally bounded.

Let $f_n(x)$ be a continuous function that is 0 for $0 \le x \le \frac{3}{2^{n+2}}$ or $\frac{3}{2^{n+1}} \le x \le 1$ and rises linearly to $f_n(x) = \frac{3}{4}$ at $x = \frac{1}{2^n}$. So $f_n \in E$.



Then $d(f_n, f_m) = \frac{3}{4}$ if $n \neq m$.

Thus if we take $\epsilon = \frac{1}{4}$, then no finite number of elements in E with balls of radius $\frac{1}{4}$ will cover $\{f_n\}$ where $n = 1,2, \dots$. This is because each ball of radius $\frac{1}{4}$ can contain at most one of the $f_n's$.

We can see this by assuming that $f_n, f_m \in B_{\frac{1}{4}}(g)$ for some $g \in C[0,1]$, and $n \neq m$.

Then by the triangle inequality we get:

$$\frac{3}{4} = d(f_n, f_m) \le d(f_n, g) + d(g, f_m) < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

which is a contradiction. Thus $B_{\frac{1}{4}}(g)$ can contain at most one f_n.

Thus no finite number of balls of radius $\frac{1}{4}$ will cover $\{f_n\}$ and thus no finite number of balls of radius $\frac{1}{4}$ will cover E.

Theorem: $f_n(x)$ converges uniformly to $f(x)$ on I if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$.

(As we will see shortly this means, if $\{f_n(x)\} \subseteq C(I)$, then $\{f_n(x)\}$ converges to $f(x) \in C(I)$, if and only if $\{f_n(x)\}$ is a Cauchy sequence in $C(I)$).

Proof: Assume that $f_n(x)$ converges uniformly to $f(x)$ on I.

By the triangle inequality we have:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$

Since $f_n(x)$ converges uniformly to $f(x)$ on I, there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for any $x \in I$.

And, of course, if $m \geq N$ then $|f_m(x) - f(x)| < \frac{\epsilon}{2}$ for any $x \in I$.

Thus if $m, n \geq N$ then we have for any $x \in I$:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now assume for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$,

if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$.

For each $x \in I$, $\{f_n(x)\}$ is a Cauchy sequence of real numbers and thus converges to a real number $f(x)$.

So $\lim_{n \to \infty} f_n(x) = f(x)$ (this is a pointwise limit).

Now we must show that $f_n(x)$ converges uniformly to $f(x)$.

By assumption, there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n, m \geq N$

then $|f_n(x) - f_m(x)| < \epsilon$.

This is true for all $m \geq N$, so let m go to ∞. So we have:

there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n, m \geq N$

then $|f_n(x) - f(x)| \leq \epsilon$.

and $f_n(x)$ converges to $f(x)$ uniformly.

Now let's see why a set of bounded uniformly convergent continuous functions must converge to a bounded continuous function. Suppose $|f_n(x)| \leq M_n$ for all $x \in I$ and each n. How do we know that as n goes to infinity, M_n doesn't go to infinity?

By the previous theorem we know that any Cauchy sequence in $C(I)$, $\{f_n(x)\}$, converges uniformly to some $f(x)$ on I (which must be continuous since all of the $f_n's$ are). Thus we have for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n \geq N$ then $|f(x) - f_n(x)| < \epsilon$.

In particular, $|f(x) - f_N(x)| < \epsilon$ for all $x \in I$. Thus we have:

$$-\epsilon < f(x) - f_N(x) < \epsilon$$

$$f_N(x) - \epsilon < f(x) < f_N(x) + \epsilon$$

$$-M_N - \epsilon \leq f_N(x) - \epsilon < f(x) < f_N(x) + \epsilon \leq M_N + \epsilon$$

Thus $|f(x)| \leq M_N + \epsilon$ and $f(x)$ is bounded.

Hence any Cauchy sequence in $C(I)$ must converge to a bounded continuous function, $f(x)$, thus $f(x) \in C(I)$ and $C(I)$ is complete.