Trigonometric Polynomials

Def. A **trigonometric polynomnial** is a function of the form:

$$T(x) = a_0 + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx))$$

where a_k and b_k are real numbers.

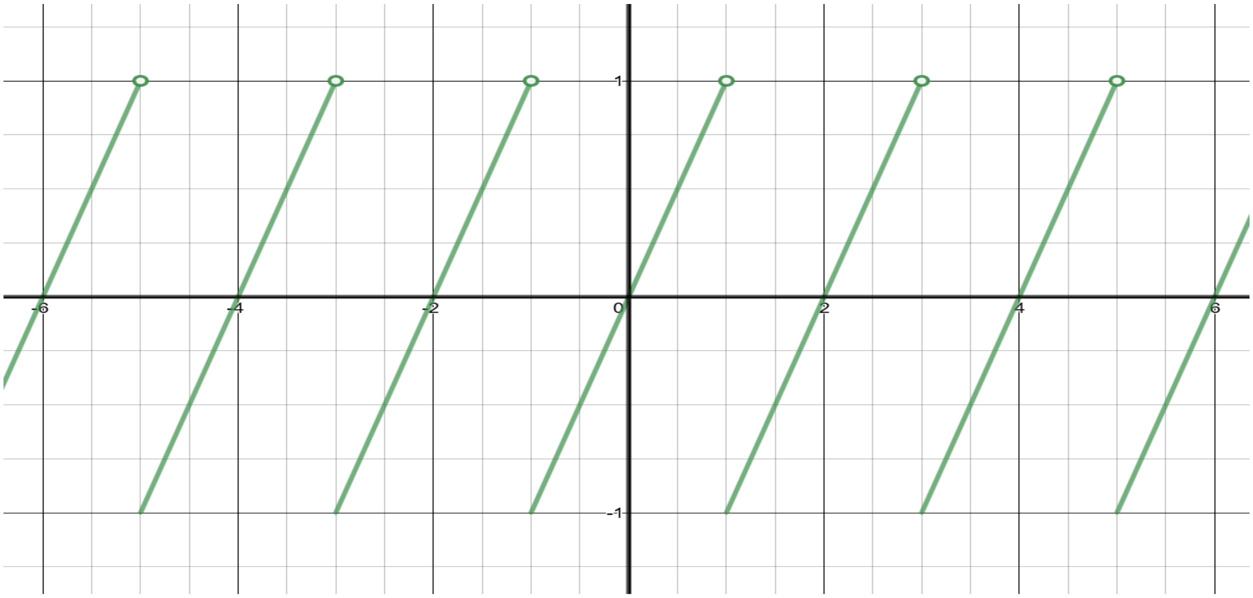
The degree of a trigonometric polynomial (trig polynomial) is the order, k, of the highest nonzero coefficient.

When working with trig polynomials it is useful to remember that :

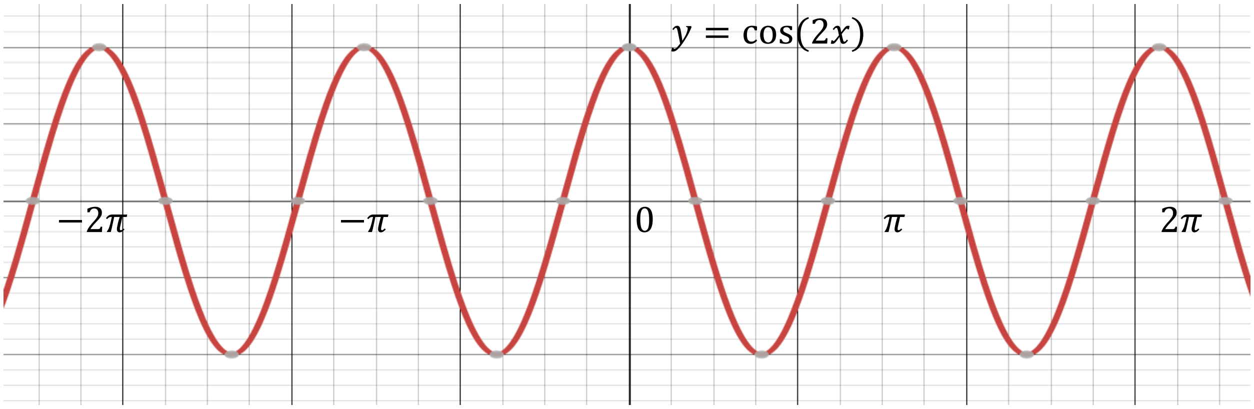
$$\sin(-x) = -\sin(x) \quad \text{and} \quad cos(-x) = \cos(x).$$

That is, $\sin(x)$ is an odd function and $\cos(x)$ is an even function.

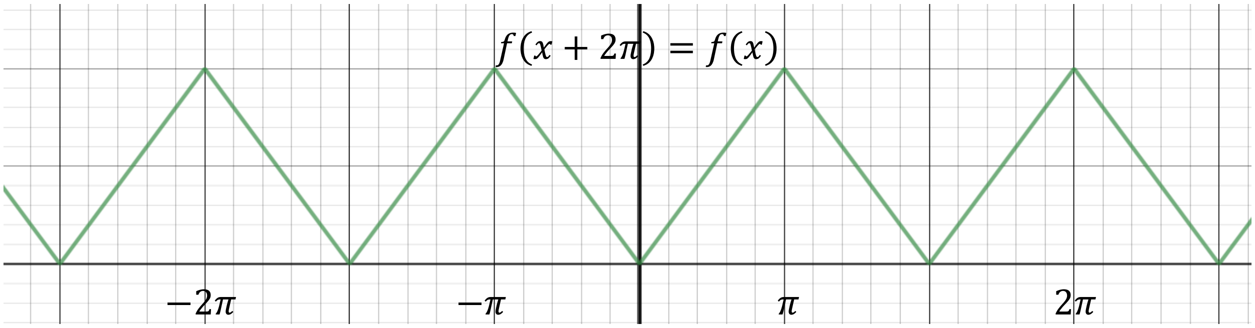
Def. we say a function, $f(x)$, is **periodic of period p**, if $f(x+p) = f(x)$ for all $x \in \mathbb{R}$, and p is the smallest such number where that is true.



Ex. $f(x) = \cos(2x)$ has a period of π.



Def. $C^{2\pi} = \{continuous\ functions\ on\ \mathbb{R}\ such\ that\ f(x + 2\pi) = f(x), x \in \mathbb{R}\}$.



Notice that every trig polynomial belongs to $C^{2\pi}$.

$C^{2\pi}$ is a vector space and a metric subspace of $C(\mathbb{R})$, bounded continuous functions on $\mathbb{R}$. $C^{2\pi}$ is complete with respect to the metric given by
$d(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$.

Our goal is to prove an anologue to the Weierstrass approximation theorem for functions in $C^{2\pi}$.

Weierstrass's Second Theorem: Given $f \in C^{2\pi}$ and $\epsilon > 0$, there is a trig polynomial T such that $\|f - T\|_\infty < \epsilon$ (i.e. $\sup_{x\in\mathbb{R}} |f(x) - T(x)| < \epsilon$). Hence, there is a sequence of trig polynomials T_n such that $T_n \to f$ uniformly on $\mathbb{R}$.

Def. $f_1, f_2, \ldots, f_n$ are **linearly independent** if $a_1 f_1 + \cdots + a_n f_n = 0$ implies that $a_1 = a_2 = \cdots = a_n = 0$.

Let $A = \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx)\}$.

We will show that the functions in A are linearly independent.

First we define an inner product (or "dot" product) on $C^{2\pi}$ by

$$\boldsymbol{< f, g >= \int_{-\pi}^{\pi} f(x) g(x) dx}.$$

We say that two elements, $f, g \in C^{2\pi}$ are **orthogonal** (or perpendicular) if

$$< f, g >= \int_{-\pi}^{\pi} f(x) g(x) dx = 0.$$

Ex. If $f(x) = 1$ and $g(x) = \cos(nx)$, $n = 1,2,3 \ldots$, then $f(x)$ and $g(x)$ are orthogonal.

$$< f, g >= \int_{-\pi}^{\pi} 1\big(cos(nx)\big) dx = \frac{1}{n} \sin(nx) \Big|_{x=-\pi}^{x=\pi} = 0.$$

Ex. All pairs of distinct elements in A are orthogonal. This follows from the trig identities:

$$(\sin(u))(\cos(v)) = \tfrac{1}{2}[\sin(u-v) + \sin(u+v)]$$

$$(\sin(u))(\sin(v)) = \tfrac{1}{2}[\cos(u-v) - \cos(u+v)]$$

$$(\cos(u))(\cos(v)) = \tfrac{1}{2}[\cos(u-v) + \cos(u+v)].$$

For example:

$$< \sin(mx), \cos(nx) >= \int_{-\pi}^{\pi} (\sin(mx))\big(cos(nx)\big)dx$$

$$= \tfrac{1}{2}\textstyle\int_{-\pi}^{\pi}(\sin((m-n)x)) + \big(sin((m+n)x)\big)dx$$

$$= \tfrac{1}{2}(-\tfrac{\cos(m-n)x}{m-n} - \tfrac{\cos(m+n)x}{m+n}\,|_{x=-\pi}^{x=\pi} = 0.$$

Now we can show that

$A = \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots, \cos(nx), \sin(nx)\}$ is a linearly independent set of functions.

Suppose $f(x) = a_0 + a_1\cos(x) + \cdots + a_n\cos(nx) + b_1\sin(x) + \cdots + b_n\sin(nx)$ and for some $a_0, \dots, a_n, b_1, \dots, b_n$, $f(x) = 0$ for all $x \in \mathbb{R}$.

Then we have:

$$0 =< 0,0 >=< f, f >$$

$$=< a_0 + a_1\cos(x) + \cdots + a_n\cos(nx) + b_1\sin(x) + \cdots + b_n\sin(nx),$$

$$a_0 + a_1\cos(x) + \cdots + a_n\cos(nx) + b_1\sin(x) + \cdots + b_n\sin(nx) >$$

$$= {a_0}^2 < 1,1 > +{a_1}^2 < \cos(x), \cos(x) > + \cdots + {a_n}^2 < \cos(nx), \cos(nx) >$$

$$+{b_1}^2 < \sin(x), \sin(x) > + \cdots + {b_n}^2 < \sin(nx), \sin(nx) >.$$

Since $< g, g > \geq 0$ and $< g, g > = 0$ if only if $g = 0$,

$< f, f > = 0$ implies that ${a_0}^2, \dots, {a_n}^2, {b_1}^2, \dots {b_n}^2 = 0$.

Thus $a_0, \ \ , a_n, b_1, \dots, b_n = 0$, and the elements of A are linearly independent.

$T(x) = a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$ is called a trig polynomial.

This is because $T(x)$ can be written as $p(sinx, cosx)$, where $p(x, y)$ is a polynomial in x and y. This follows from the fact that $\cos(kx)$ and $\sin(kx)$ can be written as polynomials in $\cos(x)$ and $\sin(x)$. For example:

$$\cos(2x) = 2cos^2(x) - 1$$

$$\begin{aligned}\cos(3x) &= \cos(2x + x) = (\cos(2x))(\cos(x)) - (\sin(2x))(\sin(x)) \\ &= (2cos^2(x) - 1)(\cos(x)) - (2\,(\sin(x))(\cos(x)))(\sin(x))) \\ &= 2(cos^3(x)) - \cos(x) - 2(sin^2(x))(\cos(x)) \\ &= 2(cos^3(x)) - \cos(x) - 2(1 - cos^2(x))(\cos(x)) \\ &= 4(cos^3(x)) - 3\cos(x).\end{aligned}$$

By using $\cos(kx) + \cos[(k-2)\,x] = 2\,[\cos((k-1)\,x)][cosx]$ we can write $\cos(kx)$ as a polynomial in just $\cos(x)$.

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\sin(3x) = \sin(2x + x) = \sin(x)\,(4cos^2(x) - 1)\,.$$

By using $\sin[(k+1)\,x] - \sin\,[(k-1)\,x] = 2(\cos\,(kx)(sinx))$ we can write $\sin(kx)$ as $\sin(x)$ times a polynomial of degree $(k-1)$ in $\cos(x)$.

Thus $\cos(kx)$ and $\sin(kx)$ can be written as polynomials of degree k in $\sin(x)$ and $\cos(x)$. Hence $T(x) = a_0 + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx))$ can be written as a polynomial of degree n in $\sin(x)$ and $\cos(x)$.

Conversely, any polynomial in $\sin(x)$ and $\cos(x)$ can be written in terms of $\cos^m(x)$ and $(\cos^{m-1}(x))(\sin(x))$, and in turn $\cos^m(x)$ and $(\cos^{m-1}(x))(\sin(x))$ can each be written in the form

$$a_0 + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx)) .$$

We can now use the Weierstrass approximation theorem to help prove Weierstrass's second theorem.

First we need:

Lemma: Given an even function $f \in C^{2\pi}$ and $\epsilon > 0$, there is an even trig polynomial T such that $\|f - T\|_\infty < \epsilon$.

Proof: Let $f \in C^{2\pi}$. The values of f are determined by its values on $[-\pi, \pi]$. Since f is even, its values are determined by its values on $[0, \pi]$.

Let $x = \cos^{-1} y$, where $-1 \le y \le 1$ and $0 \le x \le \pi$.

So $f(x) = f(\cos^{-1} y) = h(y)$, where h is continuous on $-1 \le y \le 1$.

By the Weierstrass approximation theorem there is a polynomial in y, $p(y)$, such that

$$\sup_{-1 \le y \le 1} |h(y) - p(y)| < \epsilon \quad \text{or equivalently} \quad \sup_{-1 \le y \le 1} |f(\cos^{-1} y) - p(y)| < \epsilon.$$

But $y = \cos(x)$ so $p(\cos(x))$ is a polynomial in $\cos(x)$ and we can find a trig polynomial $T(x) = p(\cos(x))$.

Thus: $$\sup_{0\le x\le\pi} |f(x) - T(x)| < \epsilon.$$

Since f and T are even and have $f(x+2\pi) = f(x)$ and $T(x+2\pi) = T(x)$

$$\sup_{x\in\mathbb{R}}|f(x) - T(x)| < \epsilon.$$

Now we apply this lemma to prove:

Weierstrass's second theorem: Given $f \in C^{2\pi}$ and $\epsilon > 0$, there is a trig polynomial T such that $\|f - T\|_\infty < \epsilon$ (i.e. $\sup_{x\in\mathbb{R}}|f(x) - T(x)| < \epsilon$). Hence, there is a sequence of trig polynomials T_n such that $T_n \to f$ uniformly on $\mathbb{R}$.

Proof. Given $f \in C^{2\pi}$, both

$$f(x) + f(-x) \quad \text{and} \quad \big(f(x) - f(-x)\big)\sin(x)$$

are even functions.

Thus by the previous lemma there are even trig polynomials T_1 and T_2 such that

$$f(x) + f(-x) = T_1(x) + e_1(x) \quad \text{and} \quad \big(f(x) - f(-x)\big)\sin(x) = T_2(x) + e_2(x)$$

where $\|e_1(x)\|_\infty < \frac{\epsilon}{2}$ and $\|e_2(x)\|_\infty < \frac{\epsilon}{2}$.

Multiplying the first equation by $\sin^2(x)$ and the second by $\sin(x)$ and adding them we get:

$$\big(f(x) + f(-x)\big)\sin^2 x = (\sin^2 x)\, T_1(x) + (\sin^2 x)\, e_1(x)$$

$$\underline{\big(f(x) - f(-x)\big)\sin^2 x = (\sin(x)\, T_2(x) + (\sin(x)\, e_2(x)}$$

$$2f(x)\sin^2 x = (\sin^2 x)T_1(x) + (sinx)T_2(x) + (\sin^2 x)e_1(x) + (sinx)e_2(x).$$

Dividing by 2 we get:

$$f(x)\sin^2 x = \frac{1}{2}[(\sin^2 x)T_1(x) + (sinx)T_2(x)]$$
$$+\frac{1}{2}[(\sin^2 x)e_1(x) + (sinx)e_2(x)].$$

But $\frac{1}{2}[(\sin^2 x)T_1(x) + (sinx)T_2(x)]$ is a trig polynomial, let's call it $T_3(x)$.

In addition

$$\sup_{x\in\mathbb{R}}\left|\frac{1}{2}[(\sin^2 x)e_1(x) + (sinx)e_2(x)]\right| \leq \sup_{x\in\mathbb{R}} |\frac{1}{2}(\sin^2 x)\big(e_1(x)\big)| +$$
$$\sup_{x\in\mathbb{R}}|\frac{1}{2}(sin\, x)\big(e_2(x)\big)|$$
$$< \quad \frac{\epsilon}{4} \quad + \quad \frac{\epsilon}{4} \quad = \frac{\epsilon}{2}\,.$$

So $f(x)\sin^2 x = T_3(x) + e_3(x)$; $\quad (*)$ where $\|e_3(x)\|_\infty < \frac{\epsilon}{2}$.

If $f \in C^{2\pi}$ then so is $f\left(x - \frac{\pi}{2}\right)$. So

$$f\left(x-\frac{\pi}{2}\right)\sin^2 x = T_4(x) + e_4(x); \quad \text{where } \|e_4(x)\|_\infty < \frac{\epsilon}{2}.$$

Replacing $x + \frac{\pi}{2}$ for x in the above equation we get:

$$f(x)\sin^2(x+\tfrac{\pi}{2}) = T_5(x) + e_5(x); \quad \text{where } \|e_5(x)\|_\infty < \frac{\epsilon}{2}.$$

$\sin\left(x+\frac{\pi}{2}\right) = \cos(x)$ so we get:

$$f(x)\cos^2(x) = T_5(x) + e_5(x). \qquad (**)$$

Now we add the two earlier equations ($(*)$ and $(**)$) :

$$f(x)\sin^2(x) = T_3(x) + e_3(x)$$

$$\underline{f(x)\cos^2(x) = T_5(x) + e_5(x)}$$

$$f(x) = T_6(x) + e_6(x)$$

where $\sup_{x\in\mathbb{R}}|e_6(x)| = \sup_{x\in\mathbb{R}}|e_3(x) + e_5(x)| \leq \sup_{x\in\mathbb{R}}|e_3(x)| + \sup_{x\in\mathbb{R}}|e_5(x)|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So we have:

$$\sup_{x\in\mathbb{R}}|f(x) - T_6(x)| = \sup_{x\in\mathbb{R}}|e_6(x)| < \epsilon.$$

Thus $\|f - T\|_\infty < \epsilon.$

Fourier Series

Given $f \in C^{2\pi}$ we can express it as the uniform limit of a sequence of trigonometric polynomials, $T_n(x)$, i.e., $T_n(x)$ converges uniformly to $f(x)$. Now we would like, at least in some cases, to calculate a sequence $T_n(x)$ where this is the case. Here we will calculate the **Fourier series** for $f(x)$.

We will start off writing:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx))$$

where the RHS is the Fourier series for $f(x)$. We write $\sim$ instead of $=$ because we don't know if the RHS will converge (pointwise) to the value of f at each $x \in \mathbb{R}$.

How do we calculate a_i, b_i?

If we multiply both sides by $\sin(mx)$ and integrate we get:

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$$

$$= \int_{-\pi}^{\pi} \sin(mx)\,[\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx))]dx$$

$$= \int_{-\pi}^{\pi} \frac{a_0}{2}\sin(mx)\,dx$$
$$+ \int_{-\pi}^{\pi} \sin\,(mx)\sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx))]dx$$

Now assuming for the moment that we can integrate term by term:

$$= \int_{-\pi}^{\pi} \frac{a_0}{2}\sin(mx)\,dx$$
$$+ \sum_{k=1}^{\infty}\int_{-\pi}^{\pi}(\sin(mx))(a_k\cos(kx) + b_k\sin(kx))dx$$

$$= b_m\int_{-\pi}^{\pi}\sin^2(mx)\,dx = b_m\int_{-\pi}^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos(2mx)\right)dx = b_m\pi.$$

So we have: $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx.$

Similarly we get: $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx.$

(with $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$).