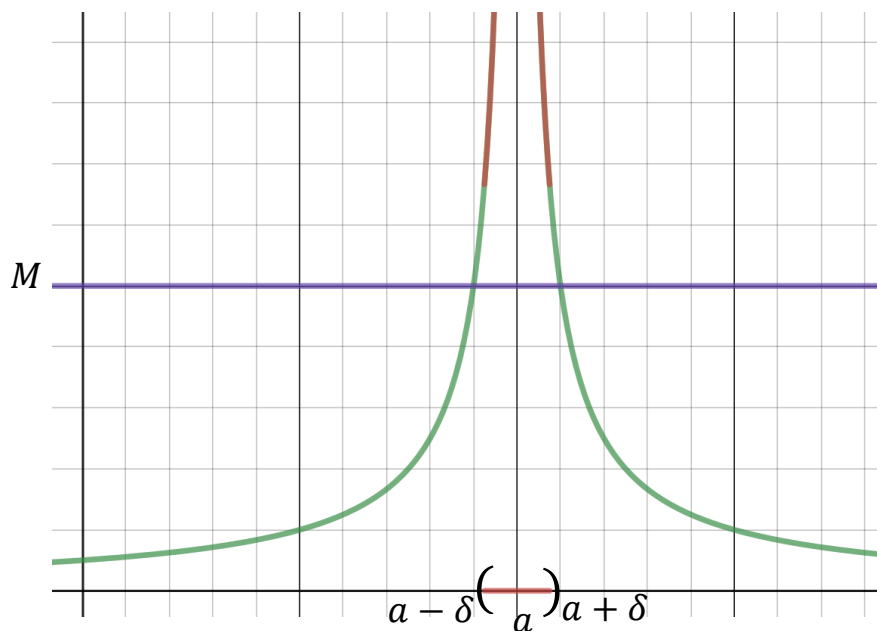
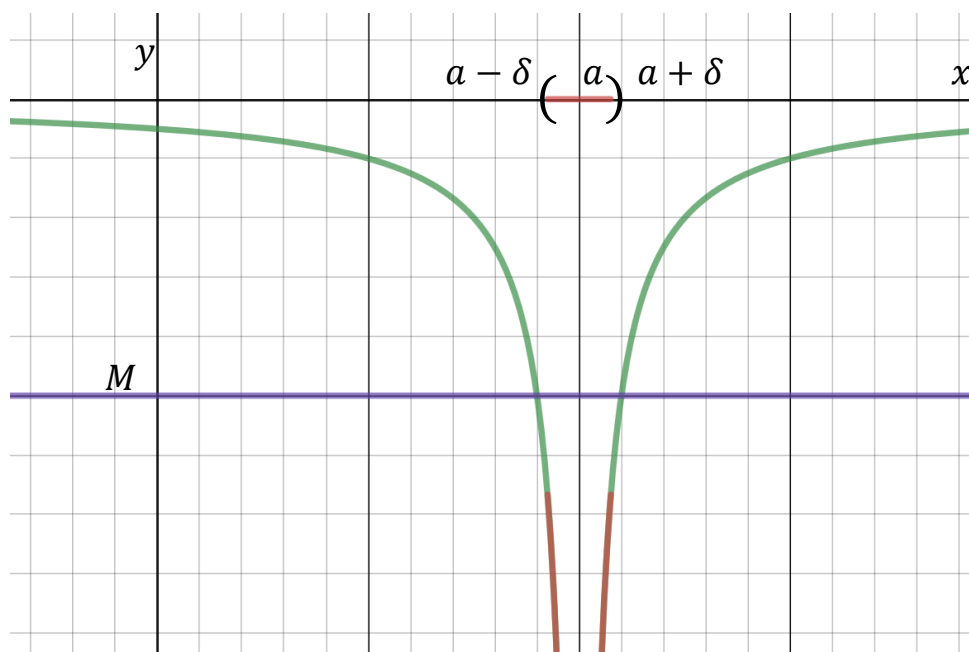


Infinite Limits and Limits at Infinity

Def. Let $f: E \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $a \in E$. $\lim_{x \rightarrow a} f(x) = +\infty$ means for every $M > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > M$.



$\lim_{x \rightarrow a} f(x) = -\infty$ means for every $M < 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) < M$.



Ex. Prove that $\lim_{x \rightarrow 3} \frac{1}{(x-3)^2} = +\infty$.

We must show given any $M > 0$ there exists a $\delta > 0$ such that if $0 < |x - 3| < \delta$ then $f(x) > M$.

Start with the statement $f(x) > M$ and work back toward the δ statement.

$\frac{1}{(x-3)^2} > M$ is equivalent to: $(x - 3)^2 < \frac{1}{M}$ since both sides are positive.

Now take square roots: $|x - 3| < \frac{1}{\sqrt{M}}$ (Note: $\sqrt{x^2} = |x|$)

Choose $\delta = \frac{1}{\sqrt{M}}$

Now let's show that this δ works:

$$|x - 3| < \delta = \frac{1}{\sqrt{M}}$$

$$|x - 3|^2 = (x - 3)^2 < \frac{1}{M}$$

$$\frac{1}{(x-3)^2} > M.$$

So we have shown $\lim_{x \rightarrow 3} \frac{1}{(x-3)^2} = +\infty$.

Ex. Prove that $\lim_{x \rightarrow 2} \frac{-1}{(x^2-4)^2} = -\infty$.

We must show given any $M < 0$ there exists a $\delta > 0$ such that if $0 < |x - 2| < \delta$ then $f(x) < M$.

Again, we start with the statement $f(x) < M$ and work backwards toward the δ statement.

$$\frac{-1}{(x^2-4)^2} < M \quad \text{is equivalent to} \quad \frac{1}{(x^2-4)^2} > -M$$

Since both sides are now positive (since $M < 0$) we have:

$$(x^2 - 4)^2 < \frac{-1}{M}; \quad \text{Factoring the LHS we get:}$$

$$(x + 2)^2(x - 2)^2 < \frac{-1}{M}.$$

Now let's find an upper bound for $(x + 2)^2$.

Choose $\delta \leq 1$.

Then $|x - 2| < 1$ or $-1 < x - 2 < 1$ now add 4 to the inequality;

$$3 < x + 2 < 5; \quad \text{now square the inequality;}$$

$$9 < (x + 2)^2 < 25; \quad \text{So now we can say that}$$

if $\delta \leq 1$ then:

$$(x + 2)^2(x - 2)^2 < 25(x - 2)^2.$$

So if we can force the RHS to be less than $\frac{-1}{M}$ we'll be in business.

$$25(x - 2)^2 < \frac{-1}{M}$$

$$(x - 2)^2 < \frac{-1}{25M}$$

$$|x - 2| < \sqrt{\frac{-1}{25M}} \quad (\text{Note: since } M < 0, \frac{-1}{25M} \text{ is a positive number}).$$

$$\text{So choose } \delta = \min \left(1, \sqrt{\frac{-1}{25M}} \right)$$

Now let's show this δ works:

If $0 < |x - 2| < \delta = \min \left(1, \sqrt{\frac{-1}{25M}} \right)$ then we have:

$$(x^2 - 4)^2 \leq 25(x - 2)^2 \quad \text{since } \delta \leq 1.$$

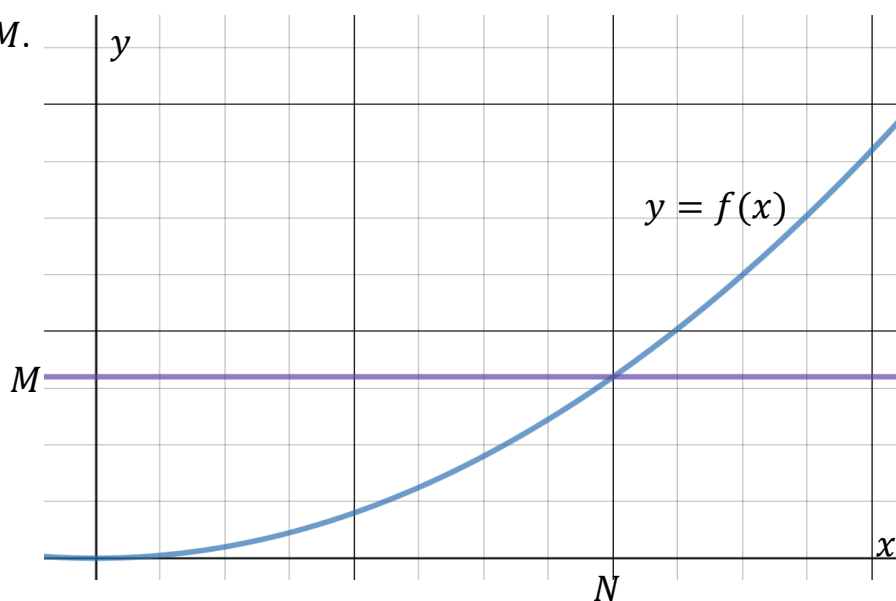
$$(x^2 - 4)^2 \leq 25(x - 2)^2 < 25\delta^2 \leq 25 \left(\frac{-1}{25M} \right) = \frac{-1}{M} \quad \text{since } \delta \leq \sqrt{\frac{-1}{25M}}.$$

$$\Rightarrow \frac{1}{(x^2 - 4)^2} > -M \quad \text{since both sides are positive; Now multiply by -1}$$

$$\frac{-1}{(x^2 - 4)^2} < M.$$

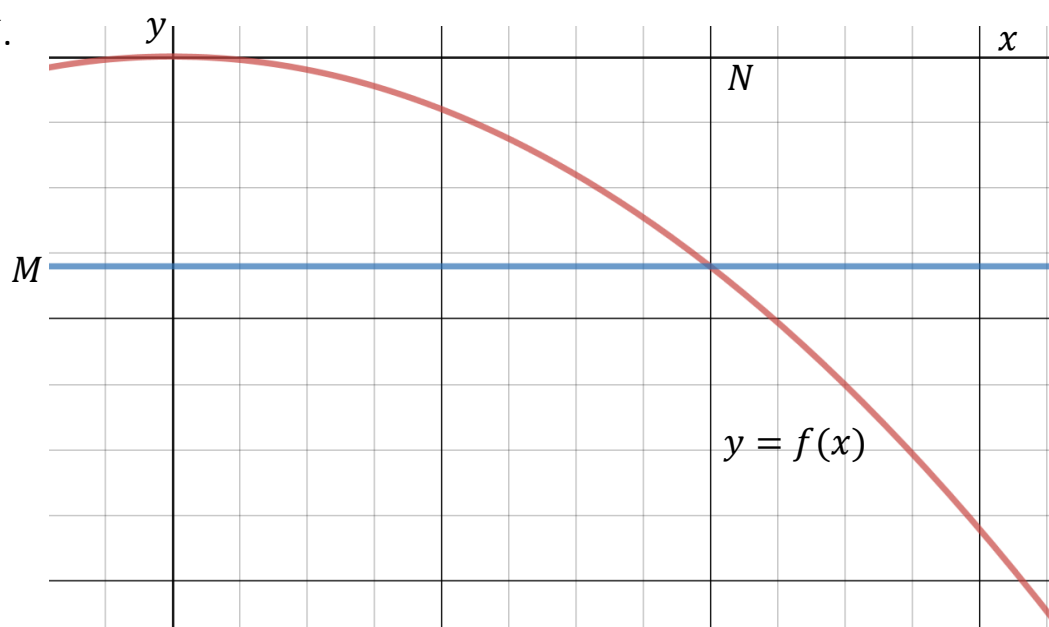
Hence we have shown: $\lim_{x \rightarrow 2} \frac{-1}{(x^2 - 4)^2} = -\infty.$

Def. Let $f: \mathbb{R} \rightarrow \mathbb{R}$. $\lim_{x \rightarrow \infty} f(x) = +\infty$ means for every $M > 0$ there exists an N such that if $x > N$ then $f(x) > M$.



$\lim_{x \rightarrow \infty} f(x) = -\infty$ means for every $M < 0$ there exists an N such that if

$x > N$ then $f(x) < M$.



The definitions of $\lim_{x \rightarrow -\infty} f(x) = +\infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$ are similar except that $x < N$.

Ex. Prove that $\lim_{x \rightarrow \infty} (x^2 - 2x) = +\infty$.

We must show that given any $M > 0$, we can find an N such that if $x > N$ then $f(x) = x^2 - 2x > M$.

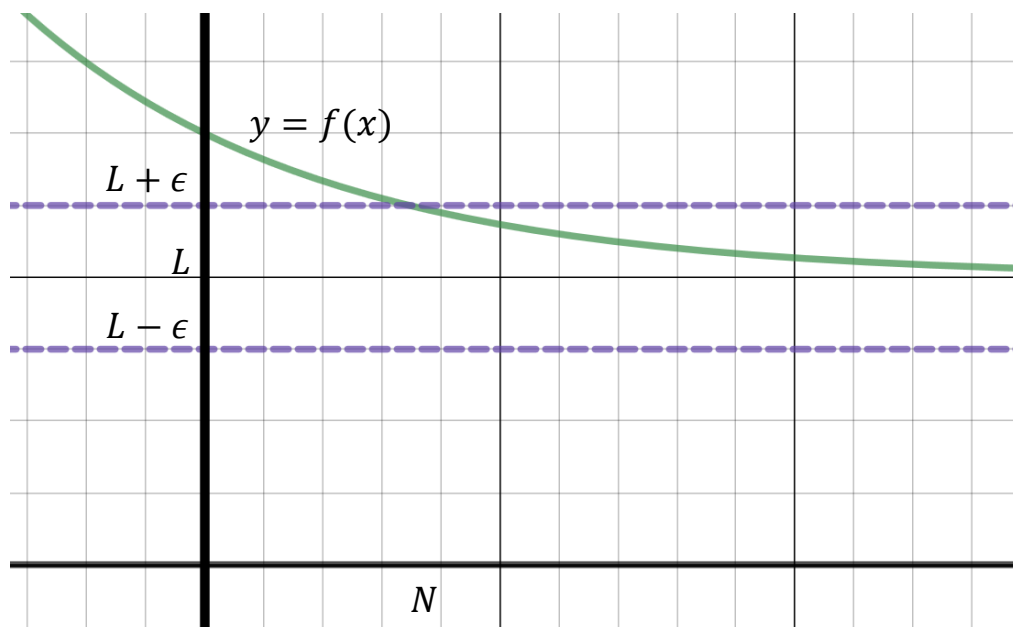
Notice that $x^2 - 2x = x(x - 2)$

So if we choose $N = M + 2$ then we have if $x > N$:

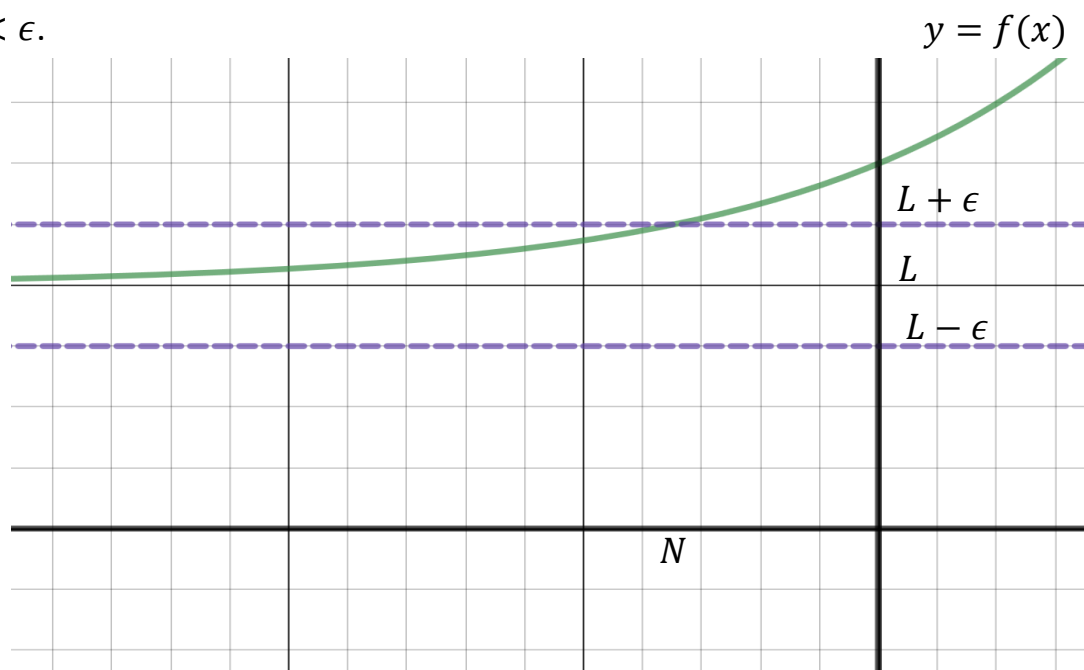
$$x(x - 2) > (M + 2)M = M^2 + 2M > M \text{ since } M^2 + M > 0 \\ \text{because } M > 0.$$

Thus we have shown $\lim_{x \rightarrow \infty} (x^2 - 2x) = +\infty$.

Def. Let $f: \mathbb{R} \rightarrow \mathbb{R}$. $\lim_{x \rightarrow \infty} f(x) = L$ means given any $\epsilon > 0$ there exists an N such that if $x > N$ then $|f(x) - L| < \epsilon$.



$\lim_{x \rightarrow -\infty} f(x) = L$ means given any $\epsilon > 0$ there exists an N such that if $x < N$ then $|f(x) - L| < \epsilon$.



Ex. Prove $\lim_{x \rightarrow -\infty} \frac{1}{x+2} = 0$.

We must show given any $\epsilon > 0$ there exists an N such that if $x < N$ then

$$\left| \frac{1}{x+2} - 0 \right| < \epsilon.$$

Start with the ϵ statement and work backwards toward the N statement.

$$\left| \frac{1}{x+2} - 0 \right| = \left| \frac{1}{x+2} \right| < \epsilon.$$

If we choose $N \leq -2$ then $\frac{1}{x+2} < 0$ for all $x < N$.

Thus in that case: $\left| \frac{1}{x+2} \right| = \frac{-1}{x+2}$.

Thus we want to force $\frac{-1}{x+2} < \epsilon$.

Now solve this inequality for x .

$$\frac{1}{x+2} > -\epsilon$$

$$x + 2 < \frac{-1}{\epsilon}$$

$$x < \frac{-1}{\epsilon} - 2.$$

Choose $N = \frac{-1}{\epsilon} - 2$ (which is also less than -2).

Let's show that this N works.

If $x < N = \frac{-1}{\epsilon} - 2$ then

$$x + 2 < \frac{-1}{\epsilon}$$

$$\frac{1}{x+2} > -\epsilon \quad \text{since both sides are negative}$$

$$\frac{-1}{x+2} < \epsilon; \quad \text{and since } x + 2 < 0, \text{ we have:}$$

$$\left| \frac{1}{x+2} - 0 \right| = \left| \frac{1}{x+2} \right| < \epsilon.$$

Thus we have shown: $\lim_{x \rightarrow -\infty} \frac{1}{x+2} = 0$.

Ex. Prove $\lim_{x \rightarrow \infty} e^{\frac{1}{x}} = 1$.

We must show that given any $\epsilon > 0$ there exists an N such that if $x > N$ then $|e^{\frac{1}{x}} - 1| < \epsilon$.

As usual, we start with the ϵ statement and work backwards toward the N statement.

Let's start by choosing $N > 0$ (the domain of the function doesn't include $x = 0$ anyway). Thus $\frac{1}{x} > 0$ and $e^{\frac{1}{x}} - 1 > 0$.

That means that $|e^{\frac{1}{x}} - 1| = e^{\frac{1}{x}} - 1 < \epsilon$; let's solve this inequality for x .

$$e^{\frac{1}{x}} < \epsilon + 1 \quad \text{Now take natural logs of both sides}$$

$$\frac{1}{x} < \ln(1 + \epsilon)$$

$$x > \frac{1}{\ln(1 + \epsilon)}, \quad \text{Since both } \frac{1}{x} > 0 \text{ and } \ln(1 + \epsilon) > 0.$$

$$\text{Choose } N = \frac{1}{\ln(1 + \epsilon)}.$$

Let's show that this N works by using the above steps in reverse:

$$\text{If } x > N = \frac{1}{\ln(1 + \epsilon)} \text{ then } x > \frac{1}{\ln(1 + \epsilon)}$$

$$\frac{1}{x} < \ln(1 + \epsilon)$$

$$e^{\frac{1}{x}} < \epsilon + 1$$

$$|e^{\frac{1}{x}} - 1| = e^{\frac{1}{x}} - 1 < \epsilon. \quad \text{Thus } \lim_{x \rightarrow \infty} e^{\frac{1}{x}} = 1.$$