Sequences- HW Problems

1. Using the ϵ-N definition of a convergent sequence, prove the following (you cannot use any theorems about convergent sequences):

a. $\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$

b. $\lim_{n\to\infty} \frac{1}{n^2} = 0$

c. $\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = 0$

d. $\lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = 0$

e. $\lim_{n\to\infty} e^{\left(\frac{n+1}{n}\right)} = e$ ($\frac{n+1}{n}$ is the exponent of e)

2. Using the definition of a convergent sequence, prove $\lim_{n\to\infty} \frac{2n}{3n+1} \neq \frac{1}{3}$.

3. Let $\{a_i\}$ be a sequence in $\mathbb{R}$. Prove $\lim_{n\to\infty} a_n = 0$, if and only if, $\lim_{n\to\infty} |a_n| = 0$.

If $\lim_{n\to\infty} |a_n| = 1$, is it true that $\lim_{n\to\infty} a_n = 1$? Prove your answer.

4. Let $\{a_i\}$ be a sequence in a metric space X,d. Prove with an ϵ-N argument that $\lim_{n\to\infty} a_n = p$ if and only if $\lim_{n\to\infty} d(a_n, p) = 0$.

5. $\{a_i\}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} a_n = 0$. Suppose that $\{b_i\}$ is a sequence in $\mathbb{R}$ such that $|b_i| \leq M$, for all i where M≥ 0. Prove from the definition of a limit of a sequence that $\lim_{n\to\infty}(a_n b_n) = 0$.

6. Suppose $\{p_i\}$ and $\{q_i\}$ are sequences of real numbers such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} q_n = q$. Give an ϵ-N proof that $\lim_{n\to\infty}(p_n + 4q_n) = p + 4q$.

7. Let $X = C[0,1]$= the set of real valued, continuous functions on $[0,1]$. Define a metric on X by $d\big(f(x), g(x)\big) = \max_{x\in[0,1]} |f(x) - g(x)|$.

   Let $\{f_n(x)\}$ be a sequence of function in X given by $f_n(x) = \frac{x}{n}$.

   Thus $f_1(x) = x$, $f_2(x) = \frac{x}{2}$, $f_3(x) = \frac{x}{3}$, $f_4(x) = \frac{x}{4}$, etc. Using

   The definition of a convergent sequence in a metric space prove that

   $\lim_{n\to\infty} f_n(x) = 0$ (i.e. $\lim_{n\to\infty} f_n(x) = f(x) = 0$).