Compact Sets and Connected Sets- HW Problems

1. Which of the sets A though I in HW problem #1 in the section called Open and Closed Sets in a Metric Space are compact?

In problems 2-5 prove the sets are compact from the definition of a compact set (you can't use any theorems about compact sets covered in class).

2. Let $E, F \subseteq X, d$ be compact subsets of a metric space X. Just using the definition of compactness (i.e., you can't use the theorem that says a union of compact sets in compact) prove the $E \cup F$ is a compact subset of X.

3. Let $E_1, E_2, \dots, E_n \subseteq X, d$ be compact subsets of a metric space X. Just using the definition of compactness prove that $E_1 \cup E_2 \cup \dots \cup E_n$, where n is a positive integer, is a compact subset of X.

4. Let p_1 and p_2 be points in a metric space X. Prove that the set $p_1 \cup p_2$ is compact in X.

5. Let $p_1, p_2, p_3, \dots, p_n$ be points in a metric space X. Prove that the set $p_1 \cup p_2 \cup p_3 \cup \dots \cup p_n$, where n is a positive integer, is compact in X.

6. Prove that the open set $(0,2)$ is not a compact subset of $\mathbb{R}$ by finding an open cover with no finite subcover.

7. Let $X = C[0,1]$= the set of real valued, continuous functions on $[0,1]$. Define a metric on X by $d\big(f(x), g(x)\big) = \max_{x\in[0,1]} |f(x) - g(x)|$. Prove that the unit ball,

$$\begin{aligned} B &= \{f(x) \in C[0,1] |\ d(f(x), 0) < 1\} \\ &= \{f(x) \in C[0,1] |\ \max_{x\in[0,1]} |f(x)| < 1\} \end{aligned}$$

is not compact. Hint: Show that B is not a closed subset of X by showing that $f(x) = 1$ is a limit point of B, but not in B.