Open and Closed Sets in a Metric Space- HW Problems

1. Make a table of the subsets of $\mathbb{R}^2$ below (using the standard metric on $\mathbb{R}^2$) where across the top you have the categories: "Limit Points", "Isolated Points", "Bounded", "Closed", and "Open", and along the left side you have the sets A through I. Identify all limit points and isolated points. Put "Y" or "N" for the rest.

$A = \{(x,y) |\, x^2 + y^2 \leq 1\}$

$B = \{(x,y) |\, 0 < x^2 + y^2 \leq 1\}$

$C = \{(x,y) |\, 0 < x^2 + y^2 < 1\}$

$D = \{(x,y) |\, \frac{1}{2} < x^2 + y^2 < 1\} \cup \{(0,0)\}$

$E = \{(x,y) |\; 0 < x < 2, \quad y = 1\}$

$F = \{(x,y) |\; 0 \leq x \leq 2, \quad 0 \leq y \leq 1\}$

$G = \{(x,y) |\; 1 \leq x, \quad 1 \leq y \leq 2\}$

$H = \{(0,0), (0,1), (1,0)\}$

$I = \{(x,y) |\; y = \sin(x)\}$

2. Prove the following (assume the standard metric on $\mathbb{R}$ and $\mathbb{R}^2$):
   a. $(-2,2)$ is an open set in $\mathbb{R}$.
   b. $[-2,2]$ is a closed set in $\mathbb{R}$.
   c. $(-2,2]$ is neither an open set nor a closed set in $\mathbb{R}$.
   d. Is $A = \{(x,y) | -2 < x < 2, \;\; y = 0\}$ open in $\mathbb{R}^2$? Prove your answer.

3. Let $A, B, C \subseteq X, d$ be non-empty open sets in a metric space X. Prove the following (without using the theorem that states that the union of open sets is open and the finite intersection of open sets is open).

   a. $A \cup B \cup C$ is open in X.

   b. $A \cap B \cap C$ is open in X.

4. Prove that If X, d is a metric space and $E \subseteq F \subseteq X$, then $\bar{E} \subseteq \bar{F}$.

5. Let $X = C[0,1]$= the set of real valued, continuous functions on $[0,1]$. Define a metric on X by $d\big(f(x), g(x)\big) = \max\limits_{x\in[0,1]} |f(x) - g(x)|$. Prove that the unit ball,

$$B = \{f(x) \in C[0,1] |\ d(f(x), 0) < 1\} = \{f(x) \in C[0,1] |\ \max_{x\in[0,1]} |f(x)| < 1\}$$

Is an open set in $C[0,1]$. Show this by taking any element $f(x) \in B$, and finding a neighborhood $N_\epsilon\big(f(x)\big)$ that lies entirely in B. That is, if you fix a function $f(x) \in B$ show you can find an ϵ such that for every $g(x) \in N_\epsilon\big(f(x)\big)$, $g(x) \in B$. Hint: If $f(x)$ is any function in B then $|f(x)|$ has a maximum value so that $\max\limits_{x\in[0,1]} |f(x)| = a < 1$.