Metric Spaces: Definitions and Examples- HW Problems

1. Prove from the definition of a metric space that $\mathbb{R}^2, d$ is a metric space where

$$d(p,q) = |p_1 - q_1| + |p_2 - q_2|; \qquad p = (p_1, p_2), \qquad q = (q_1, q_2).$$

2. Let A={positive Integers} and B={all Integers}. Let d be defined by: $d(p,q) = |p^2 - q^2|$.

   a. Prove that A, d is a metric space.

   b. Prove that B, d is not a metric space.

3. Prove that $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1|$ is not a metric on $\mathbb{R}^2$.

4a. Define a metric on $\mathbb{R}$ by $d(p,q) = |e^p - e^q|$. Find all of the points $p \in \mathbb{R}$ such that $d(p, 2) < 1$.

b. Using the metric in problem number 1, find the set of all points $p = (p_1, p_2) \in \mathbb{R}^2$ such that $d(p, O) \leq 1$, where $O = (0, 0)$. Sketch this set in $\mathbb{R}^2$.

5. Let $X = C[0,1]$= the set of real valued, continuous functions on $[0,1]$. Define 2 metrics on X by

$d_1\big(f(x), g(x)\big) = \int_0^1 |f(x) - g(x)|dx$ and

$d_2\big(f(x), g(x)\big) = \max_{x\in[0,1]} |f(x) - g(x)|$.

Let $f(x) = x$ and $g(x) = x^2$.

Find $d_1\big(f(x), g(x)\big)$, and $d_2\big(f(x), g(x)\big)$.

6. Let $X = C[0,1]$= the set of real valued, continuous functions on $[0,1]$. Define a metric on X by $d\big(f(x), g(x)\big) = \max_{x\in[0,1]} |f(x) - g(x)|$.

Determine which of the following functions is in the unit ball,

$$B = \{f(x) \in C[0,1] |\ d(f(x), 0) < 1\}.$$

In each case show why your answer is correct by either showing

$f(x) \in B$ or $f(x) \notin B$.

a. $f(x) = 3x$
b. $f(x) = \frac{1}{2}x^2$.

You can use any standard technique from first year calculus.