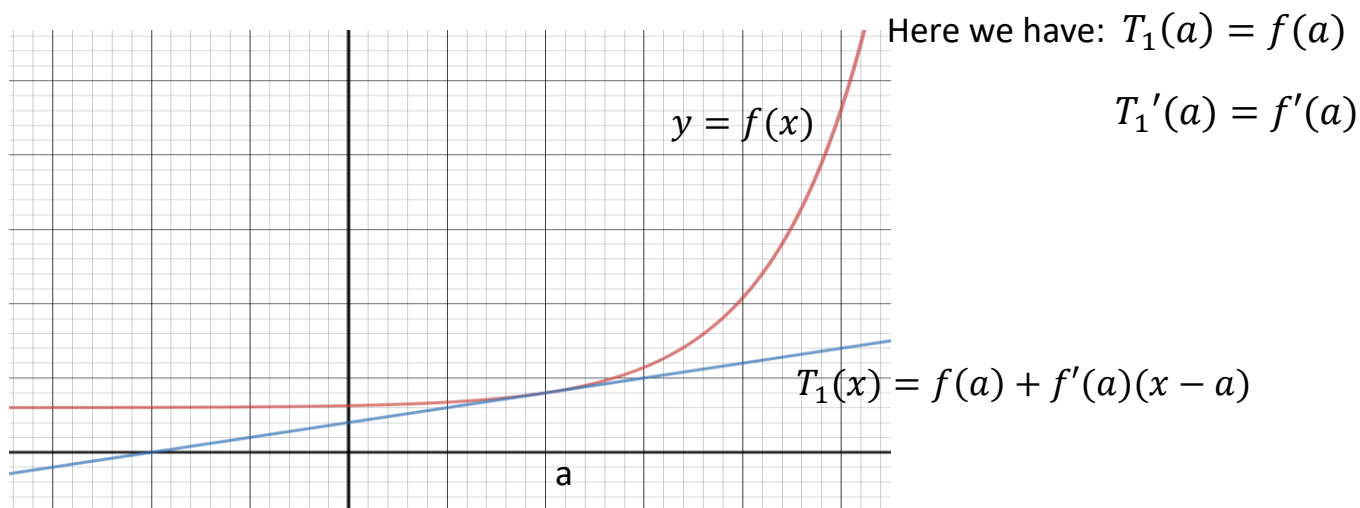


Taylor Series

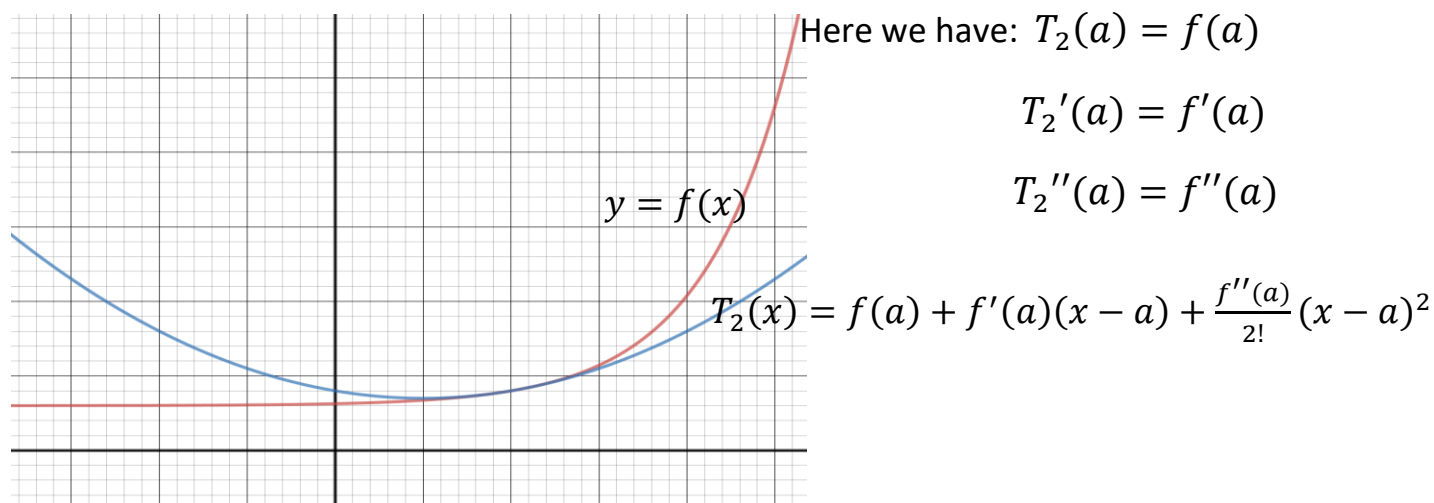
Starting with a function $f(x)$ which has infinitely many derivatives we can form a Taylor Polynomial of degree n about a point $x = a$.

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^n(a)}{n!}(x - a)^n.$$

$T_1(x) = f(a) + f'(a)(x - a)$ is a linear approximation of $f(x)$.



$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$ is a quadratic approximation of f .



$T_n(x)$ is an approximation of the function $f(x)$ which has:

$$T_n(a) = f(a), \quad T_n'(a) = f'(a), \quad T_n''(a) = f''(a), \quad \dots, \quad T_n^{(n)}(a) = f^{(n)}(a).$$

The question is, how “good” an approximation is $T_n(x)$ of $f(x)$ when $x \neq a$?

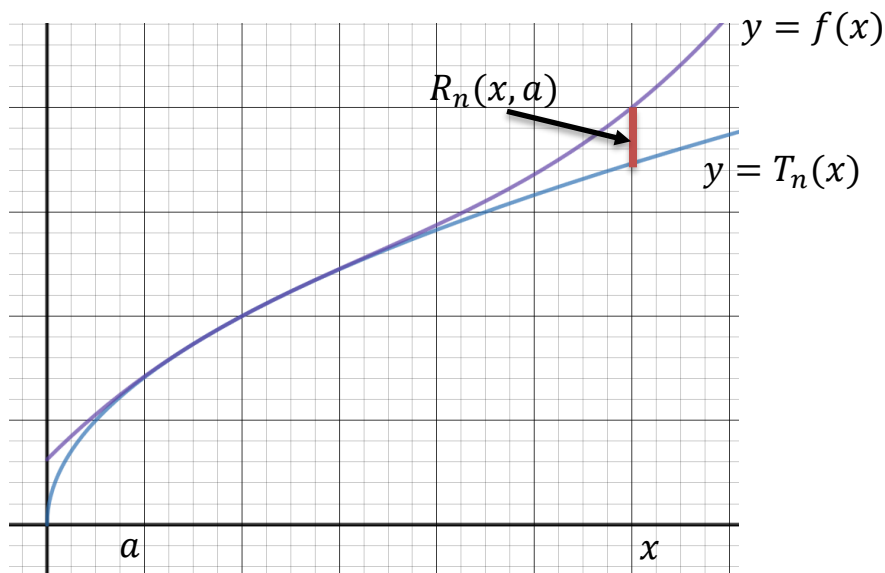
Can we put some kind of bound on how large the error is?

Theorem (Taylor’s Formula); If f has $n + 1$ derivatives in an interval I that contains “ a ”, then for $x \in I$ there is a number c , where c is strictly between x and a , such that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots$$

$$+ \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x, a).$$

where the error after the n th degree term, $R_n(x, a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$.



Note 1: “ c ” depends on x and a .

Note 2: When $n = 0$ we have:

$$f(x) = f(a) + f'(c)(x - a) \quad \text{or} \quad \frac{f(x) - f(a)}{x - a} = f'(c);$$

with c between x and a , which is just the Mean Value Theorem.

Note 3: Taylor’s formula is important because it allows us to explicitly estimate how big the error is.

Proof: We will create a function that satisfies the Mean Value Theorem and the expression $R_n(x, a) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}$, will follow from the M.V.T.

Let’s start by fixing x and a , i.e. x and a are now constants where $x \neq a$.

We define a function $g(t)$ by:

$$g(t) = f(x) - [f(t) + f'(t)(x - t) + \frac{f''(t)}{2!} (x - t)^2 + \dots + \frac{f^n(t)}{n!} (x - t)^n + R_n(x, a) \frac{(x-t)^{n+1}}{(x-a)^{n+1}}]$$

Notice that:

$$g(x) = f(x) - [f(x) + f'(x)(x - x) + \frac{f''(x)}{2!} (x - x)^2 + \dots + \frac{f^n(x)}{n!} (x - x)^n + R_n(x, a) \frac{(x-x)^{n+1}}{(x-a)^{n+1}}] = 0$$

$$\begin{aligned} g(a) &= f(x) - [f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots + \frac{f^n(a)}{n!} (x - a)^n + R_n(x, a) \frac{(x-a)^{n+1}}{(x-a)^{n+1}}] \\ &= f(x) - f(x) = 0. \end{aligned}$$

$g(t)$ satisfies the Mean Value Theorem on an interval containing x and a .

So by the Mean Value Theorem, there exists a c between a and x such that:

$$\frac{g(x)-g(a)}{x-a} = 0 = g'(c).$$

$$\begin{aligned} g'(t) = & -[f'(t) - f'(t) + f''(t)(x-t) \\ & + \frac{1}{2!} \left[(-2(x-t)f''(t) + (x-t)^2 f'''(t)) \right] + \dots \\ & + \frac{f^{(n+1)}(t)}{n!} (x-t)^n - (n+1)R_n(x, a) \frac{(x-t)^n}{(x-a)^{n+1}}] \end{aligned}$$

Which simplifies to:

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!} (x-t)^n + (n+1)R_n(x, a) \frac{(x-t)^n}{(x-a)^{n+1}}.$$

Since $g'(c) = 0$ we have:

$$0 = g'(c) = -\frac{f^{(n+1)}(c)}{n!} (x-c)^n + (n+1)R_n(x, a) \frac{(x-c)^n}{(x-a)^{n+1}}.$$

Solving for $R_n(x, a)$ we get:

$$\frac{f^{(n+1)}(c)}{n!} (x-c)^n = (n+1)R_n(x, a) \frac{(x-c)^n}{(x-a)^{n+1}}$$

$$\frac{f^{(n+1)}(c)}{(n+1)!} (x-c)^n = R_n(x, a) \frac{(x-c)^n}{(x-a)^{n+1}}$$

$$\frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} = R_n(x, a); \text{ where } c \text{ is between } x \text{ and } a.$$

Ex. Compute the Talylor polynomial $T_3(x)$ around $a = 0$ for $f(x) = \sin x$ and use it to estimate $\sin(0.1)$. Find a bound for the error in this estimate.

$$T_3(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3$$

$$f(x) = \sin x \qquad f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \qquad f'(0) = \cos 0 = 1$$

$$f''(x) = -\sin x \qquad f''(0) = -\sin 0 = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = \cos 0 = -1$$

$$f^4(x) = \sin x$$

$$T_3(x) = x - \frac{x^3}{3!}.$$

$$f(x) \approx T_3(x) = x - \frac{x^3}{3!}$$

$$f(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.099833$$

$$f(x) = \sin x = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + R_3(x, 0);$$

$$R_3(x, 0) = \frac{f^4(c)}{(4)!}(x)^4 = \frac{\sin c}{(4)!}(x)^4 ; \text{ where } c \text{ is between } 0 \text{ and } x.$$

$$|R_3(0.1, 0)| = \left| \frac{\sin c}{(4)!}(0.1)^4 \right| \leq \frac{1}{24}(0.1)^4 \approx 0.000004.$$

This means that :

$$0.099833 - 0.000004 < \sin(0.1) < 0.099833 + 0.000004 \quad \text{or} \\ 0.099829 < \sin(0.1) < 0.099837 .$$

Ex. Approximate $\ln(1.2)$ so that the error is less than 0.001.

Find the Taylor series with error term for $f(x) = \ln x$ around the point $a = 1$.

$$f(x) = \ln x$$

$$f(1) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x}$$

$$f'(1) = \frac{1}{1} = 1$$

$$f''(x) = \frac{-1}{x^2}$$

$$f''(1) = \frac{-1}{(1)^2} = -1$$

$$f'''(x) = \frac{2}{x^3}$$

$$f'''(1) = \frac{2}{(1)^3} = 2$$

$$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{x^n}$$

$$f^{(n)}(1) = (-1)^{n+1} \frac{(n-1)!}{1^n} = (-1)^{n+1} (n-1)!$$

$$T_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \\ \dots + \frac{f^{(n)}(1)}{n!}(x-1)^n .$$

$$R_n(x, 1) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-1)^{n+1} = (-1)^{n+2} \frac{(n!)}{c^{n+1}} \left(\frac{1}{(n+1)!} \right) (x-1)^{n+1} \\ = (-1)^{n+2} \frac{(x-1)^{n+1}}{(n+1)c^{n+1}} .$$

$$|R_n(x, 1)| = \left| \frac{(x-1)^{n+1}}{(n+1)c^{n+1}} \right| \quad \text{where } c \text{ is between } 1 \text{ and } x.$$

Now $x = 1.2$ so:

$$|R_n(1.2, 1)| = \left| \frac{(1.2-1)^{n+1}}{(n+1)c^{n+1}} \right| = \frac{(0.2)^{n+1}}{(n+1)c^{n+1}}.$$

Since c is between 1 and 1.2:

$$|R_n(1.2, 1)| < \frac{(0.2)^{n+1}}{(n+1)} \quad \text{and we want this to be less than } 0.001.$$

$$\text{So we must solve for } n: \quad \frac{(0.2)^{n+1}}{(n+1)} < 0.001.$$

There's no elementary way to do this, but we can just use trial and error. Just try $n = 1, 2, 3, \dots$ until we find an n that works. $n = 3$ will do the trick.

Thus we can say:

$$\ln(1.2) \approx f(1) + f'(1)(1.2 - 1) + \frac{f''(1)}{2!}(1.2 - 1)^2 + \frac{f'''(1)}{3!}(1.2 - 1)^3$$

with an error less than 0.001.

$$\ln(1.2) \approx 0 + (1.2 - 1) - \frac{1}{2}(1 - 1.2)^2 + \frac{2}{6}(1 - 1.2)^3 \approx 0.1827$$

with an error less than 0.001.

So we know that :

$$0.1827 - 0.001 < \ln(1.2) < 0.1827 + 0.001$$

$$0.1817 < \ln(1.2) < 0.1837.$$

Ex. Find the negative values for x where $f(x) = e^x$ can be approximated by $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ with an error less than 0.001.

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \text{ around } a = 0.$$

$R_3(x, 0) = \frac{f^{(4)}(c)}{(4)!} (x)^4 = \frac{e^c}{24} x^4$ we want to know the x 's such that $x < 0$ and $|R_3(x, 0)| < 0.001$.

Since $x < 0$ and $a = 0$, c , which is between x and a , is also less than 0. Thus $e^c < 1$.

$$|R_3(x, 0)| = \left| \frac{e^c}{24} x^4 \right| < \left| \frac{x^4}{24} \right| < 0.001 ; \text{ Now let's solve } x^4 < 0.024$$

$$|x| < \sqrt[4]{0.024} \approx 0.3936$$

$$\text{So } -0.3936 < x < 0.$$

Suppose $f(x)$ has infinitely many derivatives for $x \in \mathbb{R}$. When is

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$? That is, when does the Taylor Series of a function converge to the values of the function?

Theorem: If $\lim_{n \rightarrow \infty} R_n(x, a) = 0$ for $|x - a| < M$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n \text{ for all } x \text{ such that } |x - a| < M.$$

Proof: $f(x) = T_n(x) + R_n(x, a)$

$$\begin{aligned} f(x) &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots \\ &\quad + \frac{f^n(a)}{n!} (x - a)^n + R_n(x, a). \end{aligned}$$

$$\lim_{n \rightarrow \infty} f(x) = \lim_{n \rightarrow \infty} \left[\sum_{i=0}^n \frac{f^i(a)}{i!} (x - a)^i + R_n(x, a) \right]$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n ; \text{ since } \lim_{n \rightarrow \infty} R_n(x, a) = 0.$$

Ex. Prove that the Taylor Series around $a = 0$ for $f(x) = e^x$ converges to $f(x) = e^x$ for all $x \in \mathbb{R}$.

$$f(x) = e^x \qquad f(0) = e^0 = 1$$

$$f'(x) = e^x \qquad f'(0) = 1$$

$$f''(x) = e^x \qquad f''(0) = 1$$

$$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$$

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$|R_n(x, 0)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x)^{n+1} \right| = \left| \frac{e^c}{(n+1)!} (x)^{n+1} \right|;$$

where c is between 0 and x .

We need to show that $\lim_{n \rightarrow \infty} \left| \frac{e^c}{(n+1)!} (x)^{n+1} \right| = 0$ for any $x \in \mathbb{R}$.

Thus we just have to show for any fixed number x , $\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, since e^c is just a constant once x is fixed.

Fix x and let $p = \lfloor |x| \rfloor$ = the greatest integer less than or equal to $|x|$.

Notice that:

$$\frac{|x|^n}{n!} = \left(\frac{|x|}{1} \right) \left(\frac{|x|}{2} \right) \cdots \left(\frac{|x|}{p} \right) \left(\frac{|x|}{p+1} \right) \cdots \left(\frac{|x|}{n} \right) \leq \left(\frac{|x|^p}{p!} \right) \left(\frac{|x|}{p+1} \right)^{n-p}; \text{ where } \frac{|x|}{p+1} < 1.$$

We now just need to show that $\lim_{n \rightarrow \infty} \left(\frac{|x|}{p+1} \right)^{n-p} = 0$ since

$$0 \leq \frac{|x|^n}{n!} \leq \left(\frac{|x|^p}{p!} \right) \left(\frac{|x|}{p+1} \right)^{n-p}.$$

Thus by the squeeze theorem if $\lim_{n \rightarrow \infty} \left(\frac{|x|^p}{p!} \right) \left(\frac{|x|}{p+1} \right)^{n-p} = 0$ then

$$\lim_{n \rightarrow \infty} \frac{|x|^n}{n!} = 0.$$

Since $\frac{|x|}{p+1} < 1$, and $\frac{|x|^p}{p!}$ is a constant, if we can just show that $\lim_{n \rightarrow \infty} \alpha^n = 0$ if $|\alpha| < 1$, we will be done.

We must show given any $\epsilon > 0$ we can find $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $|\alpha^n - 0| < \epsilon$.

If $\alpha = 0$ then $|\alpha^n - 0| = |0 - 0| = 0 < \epsilon$; for any $n \geq 1$.

If $\alpha \neq 0$ then:

$$|\alpha|^n < \epsilon$$

$$n(\ln |\alpha|) < \ln(\epsilon)$$

$$n > \frac{\ln(\epsilon)}{\ln|\alpha|} \quad \text{since } \ln |\alpha| < 0.$$

If $\epsilon > 1$ then $\frac{\ln(\epsilon)}{\ln|\alpha|} < 0$ so let's choose $N > \max(0, \frac{\ln(\epsilon)}{\ln|\alpha|})$.

Now let's show this N works.

If $n \geq N > \max(0, \frac{\ln(\epsilon)}{\ln|\alpha|})$ then

$$\begin{aligned} |\alpha^n - 0| &= |\alpha|^n < |\alpha|^{\frac{\ln(\epsilon)}{\ln|\alpha|}} = (e^{\ln|\alpha|})^{\frac{\ln(\epsilon)}{\ln|\alpha|}} \\ &= e^{\ln|\epsilon|} = \epsilon. \end{aligned}$$

So $\lim_{n \rightarrow \infty} \frac{(|x|)^{n+1}}{(n+1)!} = 0$ and $\lim_{n \rightarrow \infty} R_n(x, 0) = 0$.

$$\text{So } f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Ex. Prove that the Taylor Series for $f(x) = \sin x$ around $a = 0$ converges to $f(x) = \sin x$ for all $x \in \mathbb{R}$.

$$f(x) = \sin x \quad f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \quad f'(0) = \cos 0 = 1$$

$$f''(x) = -\sin x \quad f''(0) = -\sin 0 = 0$$

$$f^{(3)}(x) = -\cos x \quad f^{(3)}(0) = -\cos 0 = -1$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = \sin 0 = 0.$$

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$R_{2n+1}(x, 0) = \frac{f^{(2n+2)}(c)}{(2n+2)!} (x)^{2n+2}.$$

We must show that $\lim_{n \rightarrow \infty} R_{2n+1}(x, 0) = 0$, for all $x \in \mathbb{R}$.

$$|R_{2n+1}(x, 0)| = \left| \frac{f^{(2n+2)}(c)}{(2n+2)!} (x)^{2n+2} \right|.$$

Notice that all of the derivatives of $f(x) = \sin x$ are $\pm \sin x$ or $\pm \cos x$. In any of those cases we know that $|f^{(n)}(x)| \leq 1$, for all x . Thus we know:

$$|R_{2n+1}(x, 0)| = \left| \frac{f^{(2n+2)}(c)}{(2n+2)!} (x)^{2n+2} \right| \leq \left| \frac{x^{2n+2}}{(2n+2)!} \right|. \quad \text{But notice that:}$$

$$0 \leq |R_{2n+1}(x, 0)| \leq \left| \frac{x^{2n+2}}{(2n+2)!} \right| \quad \text{and we just saw that } \lim_{n \rightarrow \infty} \frac{|(x)|^n}{(n)!} = 0.$$

So by the squeeze theorem $\lim_{n \rightarrow \infty} R_{2n+1}(x, 0) = 0$, for all $x \in \mathbb{R}$.

Thus the Taylor Series for $f(x) = \sin x$ converges to $f(x) = \sin x$ for all $x \in \mathbb{R}$.

Ex. Let $f(x) = \ln(1 - x)$.

- Find $T_3(x)$ around $a=0$.
- Approximate $\ln(1.2)$ using $T_3(x)$ around $a = 0$.
- Find an upper bound for the error in the approximation in part b.
- Prove the Taylor series for $\ln(1 - x)$, $-\sum_{n=1}^{\infty} \frac{x^n}{n}$, converges to $\ln(1 - x)$ for $-1 < x < 0$.

$$a. \quad T_3(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 \quad (\text{around } a = 0)$$

$$f(x) = \ln(1 - x) \qquad f(0) = \ln 1 = 0$$

$$f'(x) = -\frac{1}{1-x} \qquad f'(0) = -1$$

$$f''(x) = -\frac{1}{(1-x)^2} \qquad f''(0) = -1$$

$$f'''(x) = -\frac{2}{(1-x)^3} \qquad f'''(0) = -2$$

So we have:

$$T_3(x) = 0 - x - \frac{x^2}{2} - \frac{x^3}{3} = -x - \frac{x^2}{2} - \frac{x^3}{3}.$$

$$b. \quad \ln(1 - x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3}; \quad \text{so at } x = -0.2 \text{ we have:}$$

$$\ln(1.2) \approx -(-0.2) - \frac{(-0.2)^2}{2} - \frac{(-0.2)^3}{3} \approx 0.1827.$$

c. The remainder, or error term, for $T_n(x)$ around $a = 0$ is:

$$R_n(x, 0) = \frac{f^{(n+1)}(c)}{(n+1)!} (x)^{n+1}. \quad \text{In this case, } n = 3.$$

$$R_3(x, 0) = \frac{f^{(4)}(c)}{(4)!} (x)^4; \quad \text{where } c \text{ is between } x \text{ and } 0.$$

Thus the error at $x = -0.2$ is:

$$R_3(-0.2, 0) = \frac{f^{(4)}(c)}{(4)!} (-0.2)^4; \quad \text{where } -0.2 < c < 0.$$

$$f^{(4)}(x) = -\frac{6}{(1-x)^4}; \quad \text{so } f^{(4)}(c) = -\frac{6}{(1-c)^4}$$

$$R_3(-0.2, 0) = \frac{1}{(4)!} \left(-\frac{6}{(1-c)^4}\right) (-0.2)^4; \quad \text{where } -0.2 < c < 0$$

$$|R_3(-0.2, 0)| = \left| \frac{1}{(4)!} \left(-\frac{6}{(1-c)^4}\right) (-0.2)^4 \right| = \frac{(0.2)^4}{4(1-c)^4}; \quad -0.2 < c < 0.$$

Since $c < 0$, $1 - c > 1$; so we can say:

$$|R_3(-0.2, 0)| = \frac{(0.2)^4}{4(1-c)^4} < \frac{(0.2)^4}{4} = 0.0004.$$

d. To prove the Taylor series converges to the function for all $-1 < x < 0$ we must show that $\lim_{n \rightarrow \infty} R_n(x, 0) = 0$ for all $-1 < x < 0$.

$$R_n(x, 0) = \frac{f^{(n+1)}(c)}{(n+1)!} (x)^{n+1}; \quad \text{where } c \text{ is between } x \text{ and } 0.$$

$$f^{(n+1)}(x) = -\frac{n!}{(1-x)^{n+1}}; \quad \text{therefore:} \quad f^{(n+1)}(c) = -\frac{n!}{(1-c)^{n+1}}.$$

Thus we have:

$$R_n(x, 0) = \frac{-1}{(n+1)!} \left(\frac{n!}{(1-c)^{n+1}} \right) (x)^{n+1} = \frac{-1}{(n+1)} \left(\frac{1}{(1-c)^{n+1}} \right) (x)^{n+1}.$$

Notice that we can rewrite this as:

$$R_n(x, 0) = \frac{-1}{(n+1)} \left(\frac{x}{1-c} \right)^{n+1};$$

$$0 \leq |R_n(x, 0)| = \frac{1}{(n+1)} \left| \left(\frac{x}{1-c} \right) \right|^{n+1}.$$

Since $-1 < x < 0$ and $1 - c > 1$; we have $\left| \frac{x}{1-c} \right| < 1$.

If we let $\alpha = \left| \frac{x}{1-c} \right| < 1$, we know that $\lim_{n \rightarrow \infty} \alpha^n = 0$.

Thus by the squeeze theorem we can conclude that $\lim_{n \rightarrow \infty} |R_n(x, 0)| = 0$ and thus $\lim_{n \rightarrow \infty} R_n(x, 0) = 0$ for all $-1 < x < 0$.

Ex. Let $f(x) = e^{-\left(\frac{1}{x^2}\right)}$ for $x > 0$
 $= 0$ for $x \leq 0$

Show that the Taylor Series for $f(x)$ around $a = 0$ does not converge to $f(x)$ for $x > 0$.

As we noted in the last example in the section on Differentiation, for this function $f(0) = f'(0) = \dots = f^n(0) = 0$. Thus the Taylor series around $a = 0$ is:

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{n!}f^n(0)x^n + \dots$$

$$= 0.$$

Clearly, $f(x) = e^{-\left(\frac{1}{x^2}\right)}$ for $x > 0$
 $= 0$ for $x \leq 0$

Is not equal to 0 for $x > 0$. So the Taylor series does not converge to the function for $x > 0$ even though the function has an infinite number of derivatives.