The Mean Value Theorem- HW Problems

1a. Use the mean value theorem to prove that for $0 < a < b$ we have:

$$\frac{b-a}{2\sqrt{b}} < \sqrt{b} - \sqrt{a} < \frac{b-a}{2\sqrt{a}}\ .$$

b. Use part "a" to prove that: $8 + \frac{1}{9} < \sqrt{66} < 8 + \frac{1}{8}$

2a. Use the mean value theorem to prove that for $0 < a < b$ we have:

$$\left(1 - \frac{a}{b}\right) < \ln\left(\frac{b}{a}\right) < (\frac{b}{a} - 1).$$

b. Use part "a" to prove that: $\frac{1}{6} < \ln(1.2) < \frac{1}{5}$.

3. Prove $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$ by using the mean value theorem (hint: apply the mean value theorem to $f(x) = \sqrt{1+x}$ on the interval $[0, x]$).

4. $x = a$ is called a fixed point of a function $f(x)$ if $f(a) = a$ (e.g., $a = 0$ and $a = 1$ are fixed points of $f(x) = x^2$, because $f(0) = 0$ and $f(1) = 1$). Prove that if $f(x)$ is a differentiable function everywhere and $f'(x) \neq 1$ for any value of x, then $f(x)$ has at most 1 fixed point (Hint: Assume $f(x)$ has 2 fixed points $x = a$ and $x = b$. What does the mean value theorem say about the interval [a,b]?)

5. Suppose $f(x)$ is differentiable everywhere and $f(1) = -2$ and $|f'(x)| \leq 4$. Show that $-10 \leq f(3) \leq 6$ and $-4 \leq f(\frac{1}{2}) \leq 0$.

6. Suppose $f(x)$ is differentiable everywhere and $f(-3) = 2$ and $f'(x) \geq -2$. Show $f(1) \geq -6$ and $f(-5) \leq 6$.

7. Suppose $f(x)$ is differentiable on the interval $A = [a, b]$ and $f'(x) \neq 0$ on A. Prove that $f(x)$ is one-to-one on A. Hint: suppose that $f(x)$ is not one-to-one on A and derive a contradiction (ie show there must be at least one point where $f'(x) = 0$ in A).