Continuity and Compactness- HW Problems

1. In each case either prove whether the function takes on its maximum value or its minimum value on the given domains with a compactness argument or a method from 1st year calculus or demonstrate that it doesn't.

a. $f(x) = x^3$; A=[0,1]; B=(0,1)

b. $f(x) = cosx$; $A = [0,2\pi]$, $B = \left(0, \frac{3\pi}{2}\right)$, $C = (0,2\pi)$,
$D = (\frac{\pi}{2}, \frac{5\pi}{2})$.

2. Using the ϵ, δ definition of uniform continuity:

a. Prove that $f(x) = x^3$ is uniformly continuous on $(-2,2)$. (Hint: recall $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$)

b. Prove that $f(x) = x^2$ is not uniformly continuous on $(0, \infty)$.

c. Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on $\left(\frac{1}{2}, \infty\right)$.

d. Prove that $f(x) = \frac{1}{1+x}$ is uniformly continuous on $(-\infty, -\frac{3}{2}]$.

e. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. See hint on problem number 6 in the HW on Continuity.

Note: although you are not being asked to show this, $f(x) = \sqrt{x}$ is actually uniformly continuous on $[0, \infty)$, since it's uniformly continuous on $[0,1]$, because $[0,1]$ is compact.