Continuity- HW Problems

1. From the δ, ϵ definition of Continuity, prove the following functions are continuous at the indicated points:

a. $f(x) = 3x + 2;$ at $x = 1$

b. $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ $when\ x \neq 0;$

$= 0$ when $x = 0$

at $x = 0.$

c. $f(x) = x^2;$ at $x = 0\ and\ x = 3$ (prove continuity at each point separately)

2. Consider the function:

$f(x) = x$ $if\ x \geq 0$

$= x + 3$ $if\ x < 0$

a. Use a δ, ϵ argument to prove that $f(x)$ is discontinuous at $x = 0.$

b. Find an open set $U \subseteq \mathbb{R}$ such that $f^{-1}(U)$ is not open and hence $f(x)$ is not continuous on $\mathbb{R}.$

3. Let $f(x) = 0$ if x is rational

$= 1$ if x is irrational

a. Prove with a δ, ϵ argument $f(x)$ is not continuous at any point $x = a$, where "a" is a real number. (You need the fact that any interval around $x = a$, contains both rational and irrational numbers whether "a" itself is rational or irrational).

b. Find a closed set E$\subseteq\mathbb{R}$ such that $f^{-1}(E)$ is not closed, and hence f(x) is not continuous on $\mathbb{R}$.

4. Let $f(x) = 0$ if x is rational

$= x$ if x is irrational

Give a δ, ϵ proof that $f(x)$ is continuous at $x = 0$.

5. Give a δ, ϵ proof that $f(x) = x^2 + 3x$ is continuous at $x = a$, where a is any real number.

6. Give a δ, ϵ proof that $f(x) = \sqrt{x}$ is continuous at $x = a$, where a is any positive real number. Hint:

$$\sqrt{x} - \sqrt{a} = [\sqrt{x} - \sqrt{a}]\left[(\sqrt{x} + \sqrt{a})/\left(\sqrt{x} + \sqrt{a}\right)\right]$$

$$= (x - a)/\left(\sqrt{x} + \sqrt{a}\right)$$

So: $\left|\sqrt{x} - \sqrt{a}\right| = |x - a|/|(\sqrt{x} + \sqrt{a})|$

$$< |(x - a)|/\left(\sqrt{a}\right),$$ since we can force $x > 0$ by choosing $\delta \leq a$.